

PHOT 504: Quantum Photonics

Quiz 1: questions & solutions

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Exam questions

Grading: Each quiz counts for 20% of your total grade. Each question is valued equally in the score calculation.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

This document contains both the problems and their solutions.

Hints given: For Hermitian $\hat{Q}$: $\hat{Q}\psi_n = q_n\psi_n \Rightarrow \exists c_n : \psi = \sum_n c_n\psi_n, \quad \langle Q \rangle = \sum_n |c_n|^2 q_n$

$$\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y, \quad \hat{L}_+ Y_l^m = \hbar \sqrt{(l-m)(l+m+1)} Y_l^{m+1}, \quad \hat{L}_- Y_l^m = \hbar \sqrt{(l+m)(l-m+1)} Y_l^{m-1}$$

For the hydrogen atom, the $1s$ (ground state) and $2p_z$ state are given by:

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \quad \psi_{210} = \frac{1}{4\sqrt{2\pi a^3}} \frac{r}{a} e^{-r/2a} \cos \theta, \quad \psi_{2,1,\pm 1} = \mp \frac{1}{8\sqrt{2\pi a^3}} \frac{r}{a} e^{-r/2a} \sin \theta e^{\pm i\phi}$$

Question 1: 3D infinite well

Consider a particle in a 3D cubic box with sidelength L , that is: $V(x, y, z) = 0$ inside the box: $0 \leq x, y, z \leq L$, and $V(x, y, z) = +\infty$ otherwise. Stationary solutions are:

$$\begin{aligned} \text{eigenstates} \quad \psi_{n_x n_y n_z}(x, y, z) &= \sqrt{\frac{8}{L^3}} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \\ \text{eigenenergies} \quad E_{n_x n_y n_z} &= \frac{\hbar^2 \pi^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2) \end{aligned}$$

- (a) Calculate the expectation value $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$ for $\psi(x, y, z) = \psi_{231}(x, y, z)$,
(b) Then consider the superposition $\psi(x, y, z) = A(\psi_{123}(x, y, z) + \psi_{312}(x, y, z) + \psi_{321}(x, y, z))$ with A a normalization constant. What is the expectation value for the energy $\langle H \rangle$?

Solution (Q1)

(a) For the expectation value $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$ we first calculate $\langle x \rangle$. In the following we use that the 3D eigenstate is a product of 1D orthonormal eigenstates of the 1D infinite well: $\psi_{231}(x, y, z) = \psi_2(x)\psi_3(y)\psi_1(z)$:

$$\begin{aligned}\langle x \rangle &= \int_V x |\psi_{231}(x, y, z)|^2 dx dy dz, \quad V \text{ is the box volume} \\ &= \int_0^L x |\psi_2(x)|^2 dx \int_0^L |\psi_3(y)|^2 dy \int_0^L |\psi_1(z)|^2 dz \\ &= \int_0^L x |\psi_2(x)|^2 dx \cdot 1 \cdot 1 \\ &= \frac{2}{L} \int_0^L x \sin^2\left(\frac{2\pi x}{L}\right) dx \\ &= \frac{2}{L} L^2 \int_0^1 u \sin^2(2\pi u) du, \text{ with } u = x/L, \text{ } du = dx/L \\ &= \frac{2}{L} L^2 \frac{1}{4} = \frac{L}{2}\end{aligned}$$

To obtain $\langle x^2 \rangle$ we repeat exactly the same steps except that x is replaced by x^2 :

$$\begin{aligned}\langle x^2 \rangle &= \int_V x^2 |\psi_{231}(x, y, z)|^2 dx dy dz, \quad V \text{ is the box volume} \\ &= \int_0^L x^2 |\psi_2(x)|^2 dx \int_0^L |\psi_3(y)|^2 dy \int_0^L |\psi_1(z)|^2 dz \\ &= \int_0^L x^2 |\psi_2(x)|^2 dx \cdot 1 \cdot 1 \\ &= \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{2\pi x}{L}\right) dx \\ &= \frac{2}{L} L^3 \int_0^1 u^2 \sin^2(2\pi u) du, \text{ with } u = x/L, \text{ } du = dx/L \\ &= \frac{2}{L} L^3 \left(\frac{1}{6} - \frac{1}{4\pi^2 2^2} \right) = L^2 \left(\frac{1}{3} - \frac{1}{8\pi^2} \right)\end{aligned}$$

The variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$ becomes:

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = L^2 \left(\frac{1}{3} - \frac{1}{8\pi^2} \right) - \frac{L^2}{4} = L^2 \left(\frac{1}{12} - \frac{1}{8\pi^2} \right)$$

Which is a positive value with distance square as unit and $\sigma < L/2$ as it should for a particle inside the box.

(b) The superposition $\psi(x, y, z) = A(\psi_{123}(x, y, z) + \psi_{312}(x, y, z) + \psi_{321}(x, y, z))$ has three nonzero coefficients $c_{n_x n_y n_z}$: $c_{123} = c_{312} = c_{321} = A$. Since the coefficient should fulfill $\sum_{n_x, n_y, n_z} |c_{n_x n_y n_z}|^2 = 1$ we have that $3|A|^2 = 1 \Rightarrow A = 1/\sqrt{3}$.

Next we make use of the expansion of $\langle \hat{H} \rangle$ in its eigenvalues (eigenenergies):

$$\langle \hat{H} \rangle = \sum_{n_x, n_y, n_z} |c_{n_x n_y n_z}|^2 E_{n_x n_y n_z} = |c_{123}|^2 E_{123} + |c_{312}|^2 E_{312} + |c_{321}|^2 E_{321} = \frac{1}{3}(E_{123} + E_{312} + E_{321})$$

Since the three eigenenergies are the same:

$$E_{123} = E_{312} = E_{321} = 14 \frac{\hbar^2 \pi^2}{2mL^2}$$

we obtain

$$\langle \hat{H} \rangle = 14 \frac{\hbar^2 \pi^2}{2mL^2}$$

Question 2: Bra-kets in finite dimensions

Assume that the Hamiltonian $\hat{H}$ of a two-level system is represented by:

$$H = \gamma \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad \text{with eigenenergies: } E_1 = 0, \quad E_2 = 2\gamma$$

(a) Calculate normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ corresponding to $E_1 = 0$ and $E_2 = 2\gamma$.

(b) Find a time-dependent expression for the state $|S(t)\rangle$ if $|S(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$.

Solution (Q2)

(a) First we double-check that the given eigenenergies are correct:

$$0 = \begin{vmatrix} \gamma - E & i\gamma \\ -i\gamma & \gamma - E \end{vmatrix} = (\gamma - E)^2 - \gamma^2 = E(E - 2\gamma)$$

resulting in the expected roots $E_1 = 0$ and $E_2 = 2\gamma$. Next, we derive the eigenvectors $|\psi_n\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\boxed{E_1 = 0}:$$

$$\begin{pmatrix} \gamma & i\gamma \\ -i\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = ix \Rightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\boxed{E_2 = 2\gamma}:$$

$$\begin{pmatrix} \gamma & i\gamma \\ -i\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\gamma x \\ 2\gamma y \end{pmatrix} \Rightarrow y = -ix \Rightarrow |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(b) For the time-dependent expression of state $|S(t)\rangle$ with $|S(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$, we first observe that:

$$|S(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

Then we add the time-dependency factors $e^{-iE_n t/\hbar}$:

$$\begin{aligned} |S(t)\rangle &= \frac{1}{\sqrt{2}} (\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_1 t/\hbar}) \\ &= \frac{1}{\sqrt{2}} (\psi_1 + \psi_2 e^{-i2\gamma t/\hbar}) \\ &= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ i \end{pmatrix} + \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i2\gamma t/\hbar} \right) \\ &= \frac{1}{\sqrt{2}} e^{-i\gamma t/\hbar} \left(\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\gamma t/\hbar} + \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\gamma t/\hbar} \right) \\ &= \frac{1}{\sqrt{2}} e^{-i\gamma t/\hbar} \begin{pmatrix} e^{i\gamma t/\hbar} + e^{-i\gamma t/\hbar} \\ i(e^{i\gamma t/\hbar} - e^{-i\gamma t/\hbar}) \end{pmatrix} \\ &= \sqrt{2} e^{-i\gamma t/\hbar} \begin{pmatrix} \cos(\gamma t/\hbar) \\ -\sin(\gamma t/\hbar) \end{pmatrix} \end{aligned}$$

Question 3: Ladder Operators for Angular Momentum Operators

Consider a hydrogen atom in state $2p_z$, that is $\psi_{nlm} = \psi_{210}$.

(a) Derive an expression (without ladder operators) for the state $\psi = A(\hat{L}_+ + \hat{L}_-)\psi_{210}$ and find the normalization constant A .

(b) What is the expectation value of $\langle \hat{L}_+ \rangle = \langle \psi_{210} | \hat{L}_+ | \psi_{210} \rangle$?

Solution (Q3)

(a) We apply the ladder operators to ψ_{210} making use of:

$$\hat{L}_+ \psi_{n,l,m} = \hbar \sqrt{(l-m)(l+m+1)} \psi_{n,l,m+1}, \quad \hat{L}_- \psi_{n,l,m} = \hbar \sqrt{(l+m)(l-m+1)} \psi_{n,l,m-1}$$

and we obtain:

$$\psi = A(\hat{L}_+ + \hat{L}_-)\psi_{210} = A\hat{L}_+\psi_{210} + A\hat{L}_-\psi_{210} = \hbar\sqrt{2}A\psi_{2,1,1} + \hbar\sqrt{2}A\psi_{2,1,-1} = \hbar\sqrt{2}A(\psi_{2,1,-1} + \psi_{2,1,1})$$

Since ψ is a superposition of two orthonormal eigenstates $\psi_{2,1,1}$ and $\psi_{2,1,-1}$, the prefactor $\hbar\sqrt{2}A = \frac{1}{\sqrt{2}} \Rightarrow A = 1/2\hbar$.

(b) The expectation value of $\langle \hat{L}_+ \rangle = \langle \psi_{210} | \hat{L}_+ | \psi_{210} \rangle$ makes only sense from a mathematical point of view as $\hat{L}_+$ is not Hermitian. Nevertheless we can calculate the value and obtain that $\langle \hat{L}_+ \rangle$ is zero:

$$\langle \hat{L}_+ \rangle = \langle \psi_{210} | \hat{L}_+ | \psi_{210} \rangle = \hbar\sqrt{2} \langle \psi_{210} | \hat{L}_+ | \psi_{211} \rangle = 0$$