PHOT 504: Quantum Photonics

Quiz 2: questions & solutions

Michaël Barbier, Fall (2025-2026)

Exam questions

Grading: Each quiz counts for 20% of your total grade. Each question is valued equally in the score calculation.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

This document contains both the problems and their solutions.

Question 1: Spin expectation values

An electron is in spin state: $\chi = \frac{1}{\sqrt{5}} \begin{pmatrix} i \\ -2 \end{pmatrix}$

(a) Calculate the expectation value $\langle S_y \rangle = \langle \chi | \hat{S}_y | \chi \rangle$ where $\hat{S}_y$ is represented by the matrix $S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

(b) Then calculate the standard deviation $\sigma_{S_y} = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2}$.

Solution (Q1)

(a) The expectation value $\langle S_y \rangle = \langle \chi | \hat{S}_y | \chi \rangle$ is given by $\chi^\dagger S_y \chi$ in matrix representation:

$$
\begin{aligned}
\langle S_y \rangle = \chi^\dagger S_y \chi &= \frac{1}{\sqrt{5}} \begin{pmatrix} -i & -2 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} i \\ -2 \end{pmatrix} \\
&= \frac{\hbar}{10} \begin{pmatrix} -i & -2 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i \\ -2 \end{pmatrix} \\
&= \frac{\hbar}{10} \begin{pmatrix} -i & -2 \end{pmatrix} \begin{pmatrix} 2i \\ -1 \end{pmatrix} \\
&= \frac{2\hbar}{5}
\end{aligned}
$$

(b) For the standard deviation $\sigma_{S_y} = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2}$ we can use $\langle S_y\rangle$ from part (a) but we need still the first term $\langle S_y^2\rangle$:

$$\langle S_y^2\rangle = \chi^\dagger S_y \chi = \chi^\dagger \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\chi$$
$$= \frac{\hbar^2}{4}\chi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\chi = \frac{\hbar^2}{4}\chi^\dagger \chi = \frac{\hbar^2}{4}$$

The standard deviation thus becomes:

$$\sigma_{S_y} = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \frac{4\hbar^2}{25}} = \frac{3}{10}\hbar$$

Question 2: Fermions in a Harmonic Oscillator

Assume two noninteracting fermions in a 1D harmonic oscillator potential (ignore spin). One particle is in state ψ_0 and the other in ψ_1. Exchange forces will adjust the expectation value for the distance between particles:

$$\langle \Delta x^2\rangle = \langle x^2\rangle_0 + \langle x^2\rangle_1 - 2\langle x\rangle_0\langle x\rangle_1 + 2|\langle x\rangle_{01}|^2 = \frac{\hbar}{2m\omega} + \frac{3\hbar}{2m\omega} + 0 + 2|\langle x\rangle_{01}|^2$$

where we filled in the expectation values $\langle x^2\rangle_n = \langle \psi_n|x^2|\psi_n\rangle = (n+1/2)\frac{\hbar}{m\omega}$ and are only left with the unknown overlap integral: $\langle x\rangle_{01} = \int_{-\infty}^{+\infty} x\psi_0\psi_1 dx$.

(a) Explain why the third term is equal to zero.
(b) Calculate the last term of the exchange forces: $2|\langle x\rangle_{01}|^2$.

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} = \alpha\, e^{-\beta^2 x^2/2} \qquad \text{with } \alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}, \quad \beta = \sqrt{\frac{m\omega}{\hbar}}$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2} = \alpha\sqrt{2}\,\beta x\, e^{-\beta^2 x^2/2}$$

Solution (Q2)

(a) The third term $2\langle x\rangle_0\langle x\rangle_1$ is equal to zero because the expectation value of any eigenstate $\psi_n(x)$ is zero for the harmonic oscillator, which has a symmetric potential and eigenstates that are either even or odd around zero. We can show this also by explicit calculation of $\langle x\rangle_0$:

$$\langle x\rangle_0 = \int_{-\infty}^{+\infty} x|\psi_0|^2\, dx = \alpha^2 \int_{-\infty}^{+\infty} xe^{-\beta^2 x^2}\, dx = 0$$

Where the last integral is zero because the integrand is an odd function of x around zero.

(b) The last term of the expectation value $2|\langle x\rangle_{01}|^2$ can be calculated:

$$\langle x \rangle_{01} = \int_{-\infty}^{+\infty} x\psi_0\psi_1 dx$$

$$= \int_{-\infty}^{+\infty} x\left(\alpha e^{-\beta^2 x^2/2}\right)\left(\alpha\sqrt{2}\beta x e^{-\beta^2 x^2/2}\right) dx$$

$$= \alpha^2\beta\sqrt{2}\int_{-\infty}^{+\infty} x^2 e^{-\beta^2 x^2} dx$$

$$= \alpha^2\beta\sqrt{2}\frac{\sqrt{\pi}}{2(\beta^2)^{3/2}}$$

$$= \frac{\alpha^2\sqrt{\pi}}{\sqrt{2}\beta^2}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\frac{\sqrt{\pi}}{\sqrt{2}}\frac{\hbar}{m\omega} = \sqrt{\frac{\hbar}{2m\omega}}$$

Therefore $2|\langle x \rangle_{01}|^2 = \frac{\hbar}{m\omega}$ and the total expected distance (squared) is positive and larger than for distinguishable particles as expected:

$$\langle \Delta x^2 \rangle = \frac{2\hbar}{m\omega} + \frac{\hbar}{m\omega}.$$

Question 3: Symmetry and transformations

Consider a particle in an infinite well: $V(x) = 0$ for $x \in [-a, a]$ and infinite otherwise.

(a) Sketch the first three eigenstates $\psi_1(x)$, $\psi_2(x)$, and $\psi_3(x)$, before and after applying the parity operator ($\Pi : x \longrightarrow -x$). Which are symmetric/anti-symmetric under parity transformation?

(b) Assume the particle has wave function: $\psi(x) = \frac{1}{5}(3\psi_1(x) + 4\psi_2(x))$. What is the expectation value of the parity operator: $\langle \Pi \rangle = \langle \psi|\Pi|\psi \rangle$?
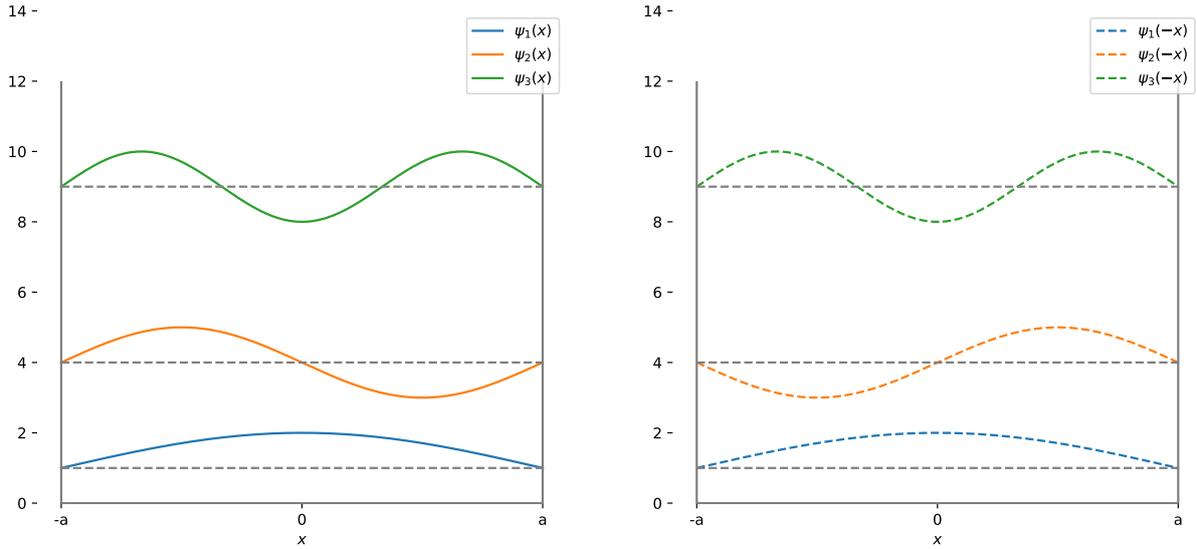
Solution (Q3)

(a) The eigenstates are even and odd function around zero, when $n = 1, 3, 5, \ldots$ then the wave function is symmetric under parity transformation, otherwise anti-symmetric:

$$\hat{\Pi}\psi_1(x) = \psi_1(-x) = \frac{1}{\sqrt{a}}\cos(\pi x/2a) = \psi(x) \qquad \text{symmetric}$$

$$\hat{\Pi}\psi_2(x) = \psi_2(-x) = -\frac{1}{\sqrt{a}}\sin(\pi x/a) = -\psi(x) \quad \text{anti-symmetric}$$

$$\hat{\Pi}\psi_3(x) = \psi_3(-x) = -\frac{1}{\sqrt{a}}\cos(3\pi x/2a) = \psi(x) \qquad \text{symmetric}$$

where $\psi_1(x) = \frac{1}{\sqrt{a}}\cos(\pi x/2a)$, $\quad \psi_2(x) = \frac{1}{\sqrt{a}}\sin(\pi x/a)$, $\quad \psi_3(x) = -\frac{1}{\sqrt{a}}\cos(3\pi x/2a)$.

(b) The expectation value $\langle \hat{\Pi} \rangle = \langle \psi | \hat{\Pi} | \psi \rangle$ for $\psi(x) = \frac{1}{5}(3\psi_1(x) + 4\psi_2(x))$ is given by:

$$
\begin{aligned}
\langle \psi | \hat{\Pi} | \psi \rangle &= \frac{1}{25} \int_{-\infty}^{+\infty} (3\psi_1 + 4\psi_2)\, \hat{\Pi}\, (3\psi_1 + 4\psi_2)\ dx \\
&= \frac{1}{25} \int_{-\infty}^{+\infty} (3\psi_1 + 4\psi_2) \left(3\hat{\Pi}\psi_1 + 4\hat{\Pi}\psi_2 \right) dx \\
&= \frac{1}{25} \int_{-\infty}^{+\infty} (3\psi_1 + 4\psi_2)(3\psi_1 - 4\psi_2)\ dx \\
&= \frac{1}{25} \int_{-\infty}^{+\infty} 9\psi_1^2 - 16\psi_2^2\ dx \\
&= \frac{1}{25} \left[9 \int_{-\infty}^{+\infty} \psi_1^2\ dx - 16 \int_{-\infty}^{+\infty} \psi_2^2\ dx \right] = -\frac{7}{25}
\end{aligned}
$$

The expectation value is in the expected interval $[-1, 1]$, and negative as the coefficient of the odd eigenstate ψ_2 is larger than the one of the symmetric eigenstate ψ_1.

As an alternative approach to reach to the expectation value we can use the expansion in eigenvalues of the parity operator. Since the Π commutes with the Hamiltonian and eigenvalues of $\hat{\Pi}\psi_n = q\psi_n$ are $q_n = (-1)^{n+1}$, we can use the expansion:

$$
\langle \Pi \rangle = \sum_n |c_n|^2 q_n = \left| \frac{3}{5} \right|^2 1 + \left| \frac{4}{5} \right|^2 (-1) = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}
$$