

# PHOT 504: Quantum Photonics

## Quiz 1: questions & solutions

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### Exam questions

**Grading:** Each quiz counts for 20% of your total grade. Each question is valued equally in the score calculation.

**Exam type:** Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

**The duration** of the quiz is 1 hour.

This document contains both the problems and their solutions.

### Question 1: Wave functions and expectation values

Consider the wave function of the ground state of the hydrogen atom:

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a},$$

with Bohr radius  $a = 0.0529$  nm.

(a) Prove that  $\psi_{100}$  is normalized.

(b) Then calculate the expectation value  $\langle r \rangle$ . You can express it in units of the Bohr radius  $a$ .

### Solution (Q1)

(a) To prove that  $\psi_{100}$  is normalized we require:

$$\iiint_{\mathbb{R}^3} |\psi_{100}|^2 dV = 1$$

Let's calculate the integral in spherical coordinates:

$$\begin{aligned}
\iiint_{\mathbb{R}^3} |\psi_{100}|^2 dV &= \int_0^{+\infty} \int_0^\pi \int_0^{2\pi} |\psi_{100}(r, \theta, \phi)|^2 r^2 \sin \theta d\theta d\phi dr \\
&= \frac{1}{\pi a^3} \int_0^{+\infty} \int_0^\pi \int_0^{2\pi} e^{-2r/a} r^2 \sin \theta d\theta d\phi dr \\
&= \frac{1}{\pi a^3} \int_0^{+\infty} r^2 e^{-2r/a} dr \cdot \int_0^\pi \sin \theta d\theta \cdot \int_0^{2\pi} d\phi \\
&= \frac{1}{\pi a^3} \cdot \frac{2!}{(2/a)^3} \cdot [-\cos \theta]_0^\pi \cdot 2\pi \\
&= \frac{1}{\pi a^3} \cdot \frac{2a^3}{8} \cdot 2 \cdot 2\pi \\
&= 1
\end{aligned}$$

The total probability is one, thus the wave function is normalized.

(b) To calculate the expectation value  $\langle r \rangle$  we calculate the above integral but with an extra factor  $r$  in the integrand:

$$\begin{aligned}
\iiint_{\mathbb{R}^3} r |\psi_{100}|^2 dV &= \int_0^{+\infty} \int_0^\pi \int_0^{2\pi} |\psi_{100}(r, \theta, \phi)|^2 r^3 \sin \theta d\theta d\phi dr \\
&= \frac{1}{\pi a^3} \int_0^{+\infty} \int_0^\pi \int_0^{2\pi} e^{-2r/a} r^3 \sin \theta d\theta d\phi dr \\
&= \frac{1}{\pi a^3} \int_0^{+\infty} r^3 e^{-2r/a} dr \cdot \int_0^\pi \sin \theta d\theta \cdot \int_0^{2\pi} d\phi \\
&= \frac{1}{\pi a^3} \cdot \frac{3!}{(2/a)^4} \cdot 4\pi = \frac{3}{2}a
\end{aligned}$$

## Question 2: Bra-kets in finite dimensions

Assume that the Hamiltonian operator  $\hat{H}$  of a two-level system is represented by the following matrix:

$$H = \begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

- (a) Solve the eigenvalue equation  $H|\psi\rangle = E|\psi\rangle$  to obtain eigenenergies  $E_n$  of the system.  
(b) Then calculate the normalized eigenstates  $|\psi_n\rangle$ .

### Solution (Q2)

(a) To solve the eigenvalue equation  $H|\psi\rangle = E|\psi\rangle$  we denote  $|\psi\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$ , and set the characteristic determinant equal to zero:

$$\begin{aligned}
0 &= \det \left[ \begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1 \end{pmatrix} - E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\
&= \det \left[ \begin{pmatrix} 1-E & i\sqrt{3} \\ -i\sqrt{3} & -(E+1) \end{pmatrix} \right] \\
&= (E+1)(E-1) - 3 = E^2 - 4 \Rightarrow E^2 = 4 \Rightarrow E_{\pm} = \pm 2
\end{aligned}$$

(b) For the normalized eigenvectors we fill in the eigenenergies  $E_{\pm}$  in the eigenvalue equation:

$$\begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = E_{\pm} \begin{pmatrix} x \\ y \end{pmatrix},$$

For  $E_1 = E_+ = 2$ :

$$\begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix},$$

Since the determinant is zero, there is one row which is superfluous and it is sufficient to use one equation:

$$x + i\sqrt{3}y = 2x \Rightarrow x = i\sqrt{3}y \Rightarrow |\psi_1\rangle = A \begin{pmatrix} i\sqrt{3} \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i\sqrt{3} \\ 1 \end{pmatrix}$$

where we find  $A = 1/2$  by setting  $\langle \psi_1 | \psi_1 \rangle = 1$ :

$$\langle \psi_1 | \psi_1 \rangle = A (-i\sqrt{3} \ 1) A \begin{pmatrix} i\sqrt{3} \\ 1 \end{pmatrix} = |A|^2 (3 + 1) = 4|A|^2 \Rightarrow A = 1/2$$

For  $E_2 = E_- = -2$ :

$$\begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} x \\ y \end{pmatrix},$$

Thus:

$$x + i\sqrt{3}y = -2x \Rightarrow y = -i\sqrt{3}x \Rightarrow |\psi_2\rangle = A \begin{pmatrix} 1 \\ -i\sqrt{3} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{3} \end{pmatrix}$$

where  $A = 1/2$  is found similar as before by setting  $\langle \psi_2 | \psi_2 \rangle = 1$ :

$$\langle \psi_2 | \psi_2 \rangle = A (1 \ i\sqrt{3}) A \begin{pmatrix} 1 \\ -i\sqrt{3} \end{pmatrix} = |A|^2 (1 + 3) = 4|A|^2 \Rightarrow A = 1/2$$

### Question 3: Harmonic oscillator

Consider a particle in a harmonic oscillator is in a superposition state:

$$|\psi\rangle = \frac{4}{5}|\psi_1\rangle - \frac{3}{5}|\psi_0\rangle$$

- (a) Derive an expression for  $\hat{x}|\psi\rangle$  using  $\hat{x}$  expressed in ladder operators  $\hat{a}_+$  and  $\hat{a}_-$ .  
 (b) What is the expectation value of the position  $\langle\hat{x}\rangle = \langle\psi|\hat{x}|\psi\rangle$ ? Use the ladder operators as in (a) to simplify the calculation.

**Hints:** the position operator  $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$ , eigenstates are orthonormal, and acting with ladder operators on eigenstates:

$$\hat{a}_+|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle, \quad \hat{a}_-|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle$$

### Solution (Q3)

- (a) Operating with the position operator gives:

$$\begin{aligned} \hat{x}|\psi\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)\left(\frac{4}{5}|\psi_1\rangle - \frac{3}{5}|\psi_0\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{4}{5}\hat{a}_+|\psi_1\rangle + \frac{4}{5}\hat{a}_-|\psi_1\rangle - \frac{3}{5}\hat{a}_+|\psi_0\rangle - \frac{3}{5}\hat{a}_-|\psi_0\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{4}{5}\sqrt{2}|\psi_2\rangle + \frac{4}{5}\sqrt{1}|\psi_0\rangle - \frac{3}{5}\sqrt{1}|\psi_1\rangle - \frac{3}{5}\sqrt{0}|\psi_0\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{4}{5}\sqrt{2}|\psi_2\rangle + \frac{4}{5}|\psi_0\rangle - \frac{3}{5}|\psi_1\rangle\right) \end{aligned}$$

- (b) To find the expectation value we can use the result of (a) and multiply on the left with the bra version of  $|\psi\rangle$ :

$$|\psi\rangle = \frac{4}{5}|\psi_1\rangle - \frac{3}{5}|\psi_0\rangle \quad \Rightarrow \quad \langle\psi| = \frac{4}{5}\langle\psi_1| - \frac{3}{5}\langle\psi_0|$$

This gives for the expectation value:

$$\begin{aligned} \langle\hat{x}\rangle &= \langle\psi|\hat{x}|\psi\rangle = \left(\frac{4}{5}\langle\psi_1| - \frac{3}{5}\langle\psi_0|\right) \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{4}{5}\sqrt{2}|\psi_2\rangle + \frac{4}{5}|\psi_0\rangle - \frac{3}{5}|\psi_1\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{4}{5}\langle\psi_1| - \frac{3}{5}\langle\psi_0|\right) \left(\frac{4}{5}\sqrt{2}|\psi_2\rangle + \frac{4}{5}|\psi_0\rangle - \frac{3}{5}|\psi_1\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left(-\frac{4}{5}\frac{3}{5}\langle\psi_1|\psi_1\rangle - \frac{3}{5}\frac{4}{5}\langle\psi_0|\psi_0\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left(-\frac{12}{25} - \frac{12}{25}\right) = -\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}} \end{aligned}$$