PHOT 504: Quantum Photonics

Final exam: questions & solutions

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Exam questions

Grading: The final exam counts for 60% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the final exam is 3 hours.

This document contains both the problems and their solutions.

Question 1: Hydrogen atom and angular momentum

Consider a **hydrogen atom** in the following superposition state $\psi = \frac{1}{\sqrt{2}}(\psi_{210} + \psi_{211})$.

(a) What are the possible values (eigenvalues of $\hat{L}_z$) and corresponding probabilities when measuring the z-component of the angular momentum L_z?
(b) Calculate the expectation value for the lowering operator $\langle \hat{L}_- \rangle$.

Solution (Q1)

(a) The hydrogen eigenstates are also eigenstates of L_z and its eigenvalues are given by $\hat{L}_z \psi_{nlm} = m\hbar\psi_{nlm}$:

$$\hat{L}_z \psi_{210} = 0\hbar\psi_{210}$$
$$\hat{L}_z \psi_{211} = 1\hbar\psi_{211}$$

Since $\hat{L}_z$ is Hermitian its eigenvalues are real and measurable: the eigenvalues are 0 and $\hbar$, and the probability to get either is the same $|c_{210}|^2 = |c_{211}|^2 = \frac{1}{2}$.

(b) The lowering operator of $\hat{L}_-$ acts on the angular part of ψ_{nlm}:

$$\hat{L}_- \psi_{nlm} = \hbar\sqrt{l(l+1) - m(m-1)}\psi_{nlm}$$

The expectation value $\langle \hat{L}_- \rangle$ is then given by:

$$\begin{aligned}\langle \hat{L}_- \rangle &= \langle \psi | \hat{L}_- | \psi \rangle \\ &= \frac{1}{2} \left(\langle 2,1,0| + \langle 2,1,1| \right) \hat{L}_- \left(|2,1,0\rangle + |2,1,1\rangle \right) \\ &= \frac{1}{2} \Big(\langle 2,1,0|\hat{L}_-|2,1,0\rangle + \langle 2,1,1|\hat{L}_-|2,1,0\rangle + \langle 2,1,0|\hat{L}_-|2,1,1\rangle + \langle 2,1,1|\hat{L}_-|2,1,1\rangle \Big) \\ &= \frac{1}{2} \Big(..\langle 2,1,0|2,1,-1\rangle + ..\langle 2,1,1|2,1,-1\rangle + \hbar\sqrt{2}\langle 2,1,0|2,1,0\rangle + ..\langle 2,1,1|2,1,0\rangle \Big) \\ &= \frac{1}{2} \Big(0 + 0 + \hbar\sqrt{2}\langle 2,1,0|2,1,0\rangle + 0 \Big) \\ &= \frac{\hbar}{\sqrt{2}}\end{aligned}$$

Where the third term is the only nonzero one due to orthonormality of the eigenstates.

Question 2: Spin in a magnetic field

Consider a spin 1/2 particle in a magnetic field oriented along the z-axis: $\vec{B} = B\vec{e}_z$, has following time-independent Pauli (Schrodinger) equation with eigenstates χ_1, χ_2 and eigenenergies E_1, E_2 are given by:

$$\mu_B B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = E \begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{cases} \text{spin-up:} & E_1 = \mu_B B, \quad \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \text{spin-down:} & E_2 = -\mu_B B, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

where $\mu_B = \frac{e\hbar}{2m_0}$ is the Bohr magneton. Assume at time zero $t = 0$ the system is in the state: $\chi(0) = \frac{1}{\sqrt{5}} (2\chi_1 + i\chi_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix}$.

(a) Derive the time-dependent probability density function: $|\chi(t)|^2$, and simplify.
(b) Write down the expression for the time-dependent expectation value $\langle \hat{S}_x \rangle = \langle \chi(t)|\hat{S}_x|\chi(t)\rangle$ and simplify.

Solution (Q2)

(a) The time-dependent probability density function $|\chi(t)|^2$ can by calculated by adding the time-dependent factors $e^{-iE_nt/\hbar}$ of the eigenstates. To simplify we denote $\omega = E_1/\hbar = -E_2/\hbar$, the time-dependent spinor becomes then:

$$\chi(t) = \begin{pmatrix} u(t) \\ d(t) \end{pmatrix} = \frac{1}{\sqrt{5}} \left(2\chi_1 e^{-iE_1 t/\hbar} + i\chi_2 e^{-iE_2 t/\hbar}\right)$$
$$= \frac{1}{\sqrt{5}} \left(2e^{-i\omega t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + ie^{i\omega t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$
$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 2e^{-i\omega t} \\ ie^{i\omega t} \end{pmatrix}$$

and thus the probability density functions of the individual spin-components u and d of $\chi = \begin{pmatrix} u \\ d \end{pmatrix}$ are independent of time and given by:

$$|u(t)|^2 = \frac{1}{5}\,|2e^{-i\omega t}|^2 = \frac{4}{5}, \qquad |d(t)|^2 = \frac{1}{5}\,|ie^{i\omega t}|^2 = \frac{1}{5}$$

And the total probability is normalized as should: $|u|^2 + |d|^2 = 1$.

(b) For the time-dependent expectation value $\langle \hat{S}_x \rangle = \langle \chi(t)|\hat{S}_x|\chi(t)\rangle$ we use the matrix representation for $\chi(t) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2e^{-i\omega t} \\ ie^{i\omega t} \end{pmatrix}$ calculated in (a), and $\hat{S}_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\langle \chi(t)|\hat{S}_x|\chi(t)\rangle = \begin{pmatrix} u^*(t) & d^*(t) \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ d(t) \end{pmatrix}$$
$$= \frac{\hbar}{2}\frac{1}{5} \begin{pmatrix} 2e^{i\omega t} & -ie^{-i\omega t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2e^{-i\omega t} \\ ie^{i\omega t} \end{pmatrix}$$
$$= \frac{\hbar}{10} \begin{pmatrix} 2e^{i\omega t} & -ie^{-i\omega t} \end{pmatrix} \begin{pmatrix} ie^{i\omega t} \\ 2e^{-i\omega t} \end{pmatrix}$$
$$= \frac{\hbar}{10}\, i2 \left(e^{i2\omega t} - e^{i2\omega t}\right)$$
$$= \frac{\hbar}{10}\, 4i^2 \sin(2\omega t) = -\frac{2\hbar}{5} \sin(2\omega t)$$

Filling in the value for the energy $\omega = \mu_B B/\hbar$ we get: $\langle \hat{S}_x \rangle = -\frac{2\hbar}{5} \sin(\frac{2\mu_B B}{\hbar} t)$.

Question 3: Perturbation of a Three-State System

Consider a three-state system: $\hat{H} = \hat{H}_0 + \hat{H}_p$ with unperturbed Hamiltonian H_0 and perturbation term H_p in matrix representation given by:

$$H = H_0 + H_p, \qquad H_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \qquad H_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(a) Find the eigenenergies ($E_1^{(0)}$, $E_2^{(0)}$, $E_3^{(0)}$), and corresponding eigenstates ($\psi_1^{(0)}$, $\psi_2^{(0)}$, $\psi_3^{(0)}$) of the unperturbed Hamiltonian by solving the eigenvalue equation $H\psi = E\psi$.
(b) Then calculate the perturbed energy values using first order perturbation theory:

$$E_n = E_n^{(0)} + E_n^{(1)}, \qquad E_n^{(1)} = \langle \psi_n^{(0)}|H_p|\psi_n^{(0)}\rangle$$

Solution (Q3)

To simplify the problem, the units of the Hamiltonian were chosen left (energy)

(a) The eigenenergies $E_1^{(0)}$, $E_2^{(0)}$, and $E_3^{(0)}$, can be found by solving the eigenvalue equation $H_0\psi = \lambda\psi$ with solutions ψ_n column vectors in $\mathbb{C}^3$.

$$(\lambda\mathbb{1} - H_0)\psi = 0 \quad \Rightarrow \quad \det(\lambda\mathbb{1} - H_0) = 0$$

We can factorize the determinant to obtain solution for λ:

$$\begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda & -2 \\ 0 & -2 & \lambda \end{vmatrix} = (\lambda+1)(\lambda^2-4) = (\lambda+1)(\lambda+2)(\lambda-2)$$

and eigenergies are thus $E_1^{(0)} = -1$, $E_2^{(0)} = -2$, $E_3^{(0)} = 2$.

For the eigenstates we get:

$\boxed{E_1^{(0)} = -1}$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = -1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \Rightarrow \quad x = x,\ y = z = 0 \quad \Rightarrow \quad \psi_1^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\boxed{E_2^{(0)} = -2}$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \Rightarrow \quad x = 0,\ z = -y \quad \Rightarrow \quad \psi_2^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\boxed{E_3^{(0)} = 2}$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \Rightarrow \quad x = 0,\ z = y \quad \Rightarrow \quad \psi_3^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(b) To find the perturbed energies up to 1th order

$$E_1^{(1)} = \langle\psi_1^{(0)}|H_p|\psi_1^{(0)}\rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$E_2^{(1)} = \langle\psi_2^{(0)}|H_p|\psi_2^{(0)}\rangle = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$E_3^{(1)} = \langle\psi_3^{(0)}|H_p|\psi_3^{(0)}\rangle = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$$

Up to first order the eigenenergies do not differ in this case: $E_1 = E_1^{(0)} = -1$, $E_2 = E_2^{(0)} = -2$, $E_3 = E_3^{(0)} = 2$.

The actual energies (**not required for this question**) can be obtained from directly solving the perturbed Hamiltonian and are different:

$$\begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-1 & -2 \\ 0 & -2 & \lambda+1 \end{vmatrix} = (\lambda+1)\left[(\lambda^2-1)-4\right] = (\lambda+1)(\lambda^2-5)$$

and the actual eigenergies are thus $E_1 = -1$, $E_2 = -\sqrt{5}$, $E_3 = \sqrt{5}$.

Question 4: Time-reversal symmetry of a spin 1/2 particle

Consider a spin 1/2 particle defined in the standard basis of spin-up and spin-down: $\{|\uparrow\rangle, |\downarrow\rangle\}$. The time-reversal operator Θ acting on a spinor has the following effect:

$$\Theta \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$$

(a) Assume the particle is in the spin-up state: $\chi_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, in what state is the particle after you apply the time-reversal operator Θ?

(b) Suppose now the particle is in state $\chi = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$. What is the expectation value $\langle\Theta\rangle = \langle\chi|\Theta|\chi\rangle$?

Solution (Q4)

(a) Applying the time-reversal operator to the spin-up state results in the spin-down state:

$$\Theta\chi_u = \Theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) We can calculate the expectation value of the time-reversal operator for a more general spinor $\chi = \begin{pmatrix} u \\ d \end{pmatrix}$:

$$\langle\Theta\rangle = \langle\chi|\Theta|\chi\rangle = \begin{pmatrix} u^* & d^* \end{pmatrix} \Theta \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} u^* & d^* \end{pmatrix} \begin{pmatrix} -d^* \\ u^* \end{pmatrix} = -u^*d^* + d^*u^* = 0$$

This result is also valid for $\chi = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and further tells us that $\Theta|\chi\rangle$ and $|\chi\rangle$ are orthogonal. btw: this is true for general spinors described by the Pauli-Schrodinger equation.

Question 5: Life time of a particle in a hydrogen atom

The life-time of a particle is given by $\tau = 1/A$. Assume a particle is in the p_z orbital, i.e. $\psi_{2,1,0}$ (ignore spin) and consider the life-time for falling back to the ground state $\psi_{1,0,0}$. In this case $\mathcal{P}_x$ and $\mathcal{P}_y$ are zero and $|\mathcal{P}|^2 = |\mathcal{P}_z|^2$. In this case A is given by:

$$A = \frac{\Delta\omega^3}{3\pi\epsilon_0\hbar c^3}|\mathcal{P}_z|^2$$

(a) First prove that $\mathcal{P}_x = q\langle\psi_{1,0,0}|x|\psi_{2,1,0}\rangle = 0$.
(b) Then calculate $|\mathcal{P}_z|^2 = q^2|\langle\psi_{1,0,0}|z|\psi_{2,1,0}\rangle|^2$.

Solution (Q5)

The eigenstates are given by (both carthesian and spherical coordinates):

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}, \qquad \psi_{100}(x,y,z) = \frac{1}{\sqrt{\pi a^3}}e^{-\sqrt{x^2+y^2+z^2}/a}$$

$$\psi_{210}(r,\theta,\phi) = \frac{1}{4\sqrt{2\pi a^3}}\frac{r}{a}e^{-r/2a}\cos\theta, \qquad \psi_{210}(x,y,z) = \frac{1}{4\sqrt{2\pi a^3}}\frac{z}{a}e^{-\sqrt{x^2+y^2+z^2}/2a}$$

(a) To prove that $\mathcal{P}_x = q\,\langle\psi_{1,0,0}|x|\psi_{2,1,0}\rangle = 0$ we can argue using symmetry around the z-axis of both $|\psi_{2,1,0}\rangle$ and $|\psi_{1,0,0}\rangle$ and the integral over the volume with positive x will compensate for the volume with negative x:

$$\begin{aligned}
\langle\psi_{1,0,0}|x|\psi_{2,1,0}\rangle &= \iiint_{\mathbb{R}^3} x\psi_{100}^*\psi_{210}\,dx\,dy\,dz \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{0} x\psi_{100}^*\psi_{210}\,dx + \int_0^{+\infty} x\psi_{100}^*\psi_{210}\,dx\right)dy\,dz \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}(-x)\psi_{100}^*(-x,y,z)\psi_{210}(-x,y,z)\,dx\right. \\
&\qquad\left. + \int_0^{+\infty} x\psi_{100}^*(x,y,z)\psi_{210}(x,y,z)\,dx\right)dy\,dz \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(-\int_0^{+\infty} x\psi_{100}^*(x,y,z)\psi_{210}(x,y,z)\,dx\right. \\
&\qquad\left. + \int_0^{+\infty} x\psi_{100}^*(x,y,z)\psi_{210}(x,y,z)\,dx\right)dy\,dz \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(0)\,dy\,dz = 0
\end{aligned}$$

Where we used the fact that ψ_{100} and ψ_{210} are real, and symmetric in x around $x = 0$:

$$\begin{aligned}
\psi_{100}^*(-x,y,z) = \psi_{100}^*(x,y,z) &= \psi_{100}(x,y,z) \\
\psi_{210}(-x,y,z) &= \psi_{210}(x,y,z)
\end{aligned}$$

Alternative solution: We can get to the same result by using spherical coordinates and $x = r\sin\theta\cos\phi$:

$$\begin{aligned}
\langle\psi_{1,0,0}|x|\psi_{2,1,0}\rangle &= \iiint_{\mathbb{R}^3} r\sin\theta\cos\phi\,\psi^*_{100}\psi_{210}\,r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= \int_0^{+\infty} r^3\frac{1}{\sqrt{\pi a^3}}e^{-r/a}\frac{1}{4\sqrt{2\pi a^3}}\frac{r}{a}e^{-r/2a}dr\ \int_0^{\pi}\cos\theta\sin^2\theta d\theta\ \int_0^{2\pi}\cos\phi\,d\phi \\
&= \frac{1}{4\sqrt{2}\pi a^4}\int_0^{+\infty} r^4e^{-3r/2a}dr\ \int_0^{\pi}\cos\theta\sin^2\theta d\theta\ \int_0^{2\pi}\cos\phi\,d\phi = 0
\end{aligned}$$

Since the last factor is zero (and the other factors are finite) the whole integral becomes zero:

$$\int_0^{2\pi}\cos\phi\,d\phi = [\sin\phi]\Big|_0^{2\pi} = 0$$

(b) To calculate the z-component of $\mathcal{P}$: $\mathcal{P}_z = q\langle\psi_{1,0,0}|z|\psi_{2,1,0}\rangle$ we use integration in spherical coordinates and use $z = r\cos\theta$:

$$\begin{aligned}
\langle\psi_{1,0,0}|z|\psi_{2,1,0}\rangle &= \iiint_{\mathbb{R}^3} r\psi^*_{100}\psi_{210}r^2\sin\theta\cos\theta\,dr\,d\theta\,d\phi \\
&= \int_0^{+\infty} r^3\frac{1}{\sqrt{\pi a^3}}e^{-r/a}\frac{1}{4\sqrt{2\pi a^3}}\frac{r}{a}e^{-r/2a}dr\cdot\int_0^{\pi}\cos^2\theta\sin\theta d\theta\cdot\int_0^{2\pi}d\phi \\
&= \frac{1}{4\sqrt{2}\pi a^4}\int_0^{+\infty} r^4e^{-3r/2a}\,dr\cdot\left[-\frac{1}{2+1}\cos^3\theta\right]\Bigg|_0^{\pi}\cdot 2\pi \\
&= \frac{1}{4\sqrt{2}\pi a^4}\frac{4!}{(3/2a)^5}\cdot\frac{2}{3}\cdot 2\pi \\
&= \frac{2^8}{3^5\sqrt{2}}\cdot a
\end{aligned}$$

Then we take the absolute value square: $|\mathcal{P}_z|^2 = \frac{2^{15}}{3^{10}}\cdot q^2a^2$.

This last part is not required for this question: This value could then be used to calculate the spontaneous emission rate A where we fill in $\Delta\omega = -\frac{3}{4}\frac{\text{Ry}}{\hbar}$ and apply once $\text{Ry} = \frac{me^4}{8\epsilon_0^2h^2}$ and fill in the Bohr radius $a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ again once:

$$\begin{aligned}
A &= \frac{\Delta\omega^3}{3\pi\epsilon_0\hbar c^3}|\mathcal{P}_z|^2 = \frac{\Delta\omega^3}{3\pi\epsilon_0\hbar c^3}\cdot\frac{2^{15}}{3^{10}}\cdot e^2a^2 = \frac{\left(\frac{3}{4}\text{Ry}\right)^3}{3\pi\epsilon_0\hbar^4c^3}\cdot\frac{2^{15}}{3^{10}}\cdot e^2a^2 = \frac{\text{Ry}^3}{\pi\epsilon_0\hbar^4c^3}\cdot\frac{2^9}{3^8}\cdot e^2a^2 \\
&= \frac{\text{Ry}^3}{\hbar^4c^3}\cdot\frac{2^9}{3^8}\cdot\frac{4\hbar^2}{m}a = \frac{\text{Ry}^2}{m^2c^4}\cdot\frac{2^{10}}{3^8}\cdot\frac{c}{a}
\end{aligned}$$

Remembering that $mc^2 = 0.511\,\text{MeV}$ we obtain:

$$A = \frac{\text{Ry}^2}{m^2c^4}\cdot\frac{2^{10}}{3^8}\cdot\frac{c}{a} = \left(\frac{13.6\,\text{eV}}{0.511\times10^6\,\text{eV}}\right)^2\cdot\frac{2^{10}}{3^8}\cdot\frac{3\times10^8\text{m/s}}{0.529\times10^{-10}\text{m}} \approx 6.27\times10^8\,\text{s}^{-1}$$

Therefore $\tau = 1/A \approx 1.6\times10^{-9}$ s