PHOT 504 Fall (2025-2026) **Final exam** Date: 22 January 2026
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Student Name:

Grading: The final exam counts for 60% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the final exam is 3 hours.

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam. Please tell if any question is unclear or ambiguous.

Question 1: Hydrogen atom and angular momentum

Consider a **hydrogen atom** in the following superposition state $\psi = \frac{1}{\sqrt{2}}\left(\psi_{210} + \psi_{211}\right)$.

(a) What are the possible values (eigenvalues of $\hat{L}_z$) and corresponding probabilities when measuring the z-component of the angular momentum L_z?
(b) Calculate the expectation value for the lowering operator $\langle \hat{L}_- \rangle$.

Question 2: Spin in a magnetic field

Consider a spin 1/2 particle in a magnetic field oriented along the z-axis: $\vec{B} = B\vec{e}_z$, has following time-independent Pauli (Schrodinger) equation with eigenstates χ_1, χ_2 and eigenenergies E_1, E_2 are given by:

$$\mu_B B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = E \begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{cases} \text{spin-up:} & E_1 = \mu_B B, \quad \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \text{spin-down:} & E_2 = -\mu_B B, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

where $\mu_B = \frac{e\hbar}{2m_0}$ is the Bohr magneton. Assume at time zero $t = 0$ the system is in the state: $\chi(0) = \frac{1}{\sqrt{5}}\left(2\chi_1 + i\chi_2\right) = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ i \end{pmatrix}$.

(a) Derive the time-dependent probability density function: $|\chi(t)|^2$, and simplify.
(b) Write down the expression for the time-dependent expectation value $\langle \hat{S}_x \rangle = \langle \chi(t) | \hat{S}_x | \chi(t) \rangle$ and simplify.

Question 3: Perturbation of a Three-State System

Consider a three-state system: $\hat{H} = \hat{H}_0 + \hat{H}_p$ with unperturbed Hamiltonian H_0 and perturbation term H_p in matrix representation given by:

$$H = H_0 + H_p, \qquad H_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \qquad H_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(a) Find the eigenenergies ($E_1^{(0)}$, $E_2^{(0)}$, $E_3^{(0)}$), and corresponding eigenstates ($\psi_1^{(0)}$, $\psi_2^{(0)}$, $\psi_3^{(0)}$) of the unperturbed Hamiltonian by solving the eigenvalue equation $H\psi = E\psi$.
(b) Then calculate the perturbed energy values using first order perturbation theory:

$$E_n = E_n^{(0)} + E_n^{(1)}, \qquad E_n^{(1)} = \langle \psi_n^{(0)} | H_p | \psi_n^{(0)} \rangle$$

Question 4: Time-reversal symmetry of a spin 1/2 particle

Consider a spin 1/2 particle defined in the standard basis of spin-up and spin-down: $\{|\uparrow\rangle, |\downarrow\rangle\}$. The time-reversal operator Θ acting on a spinor has the following effect:

$$\Theta \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$$

(a) Assume the particle is in the spin-up state: $\chi_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, in what state is the particle after you apply the time-reversal operator Θ?
(b) Suppose now the particle is in state $\chi = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$. What is the expectation value $\langle \Theta \rangle = \langle \chi | \Theta | \chi \rangle$?

Question 5: Life time of a particle in a hydrogen atom

The life-time of a particle is given by $\tau = 1/A$. Assume a particle is in the p_z orbital, i.e. $\psi_{2,1,0}$ (ignore spin) and consider the life-time for falling back to the ground state $\psi_{1,0,0}$. In this case $\mathcal{P}_x$ and $\mathcal{P}_y$ are zero and $|\mathcal{P}|^2 = |\mathcal{P}_z|^2$. In this case A is given by:

$$A = \frac{\Delta\omega^3 \hbar}{\pi^2 c^3} |\mathcal{P}_z|^2$$

(a) First prove that $\mathcal{P}_x = q\langle \psi_{1,0,0} | x | \psi_{2,1,0} \rangle = 0$.
(b) Then calculate $|\mathcal{P}_z|^2 = q^2 |\langle \psi_{1,0,0} | z | \psi_{2,1,0} \rangle|^2$.

Hints: For Hermitian $\hat{Q}$: $\hat{Q}\,\psi_n = q_n\,\psi_n \quad \Rightarrow \quad \exists c_n : \psi = \sum_n c_n \psi_n, \quad \langle Q \rangle = |c_n|^2 q_n$

$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y, \quad \hat{L}_+ Y_l^m = \hbar\sqrt{(l-m)(l+m+1)}\, Y_l^{m+1}, \quad \hat{L}_- Y_l^m = \hbar\sqrt{(l+m)(l-m+1)}\, Y_l^{m-1}$

For the hydrogen atom, the $1s$ (ground state) and $2p_z$ state are given by:

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \qquad \psi_{210} = \frac{1}{4\sqrt{2\pi a^3}} \frac{r}{a} e^{-r/2a} \cos\theta.$$