PHOT 451: Quantum Photonics

Quiz 2: questions & solutions

Michaël Barbier, Fall (2025-2026)

Exam questions

Grading: Each quiz counts for 15% of your total grade.

Exam type: Take-home exam, please fill in using your own-written descriptions/solutions and sketches. You can use a computer using but no typed text.

This document contains both the problems and their solutions.

Question 1: Diffraction limit

Consider an optical system existing of a single thin microlens (glass $n = 1.5 = 3/2$) with clear diameter $D = 10$ micron, curvature radii $R_1 = 50$ micron, $R_2 = -50$ micron, with incident light of wavelength λ. RMS OPD is given by $\sigma_w = 0.2$ in units of wavelength $\lambda = 500$ nm.

(a) Calculate the focal length f, the Airy disk diameter of the PSF, and the approximate Strehl ratio (you can use $S \approx e^{-4\pi^2\sigma_w^2}$).
(b) Exchange the lens for a smaller one with clear diameter $D = 5$ micron but with equal curvature radii: how do the following values scale qualitatively?

- Spherical aberration
- RMS OPD
- PSF airy diameter (diffraction spot size)
- Strehl ratio

Solution (Q1)

(a) The **focal length** f can be obtained from the Lensmaker's formula:

$$f = \left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]^{-1} = \left[\frac{1}{2}\frac{2}{50}\right]^{-1} \text{µm} = 50\,\text{µm}$$

Airy disk diameter d of the PSF is given by the expression:

$$d = 2.44\frac{\lambda f}{D} = 2.44\frac{0.5 \cdot 50}{10}\,\mu\text{m} = 6.1\,\mu\text{m}$$

Regarding the **Strehl ratio**: there was an error in the original question, the wrong notation was used for the phase error $\sigma_w \leftrightarrow \sigma_\phi$ where the relation between the two is $\sigma_\phi = 2\pi\sigma_w$. Therefore both $S \approx e^{-4\pi^2\sigma_w^2}$ or $S \approx e^{-\sigma_w^2}$ will be accepted as solution. The Strehl ratio as approximated by $S \approx e^{-4\pi^2\sigma_w^2}$:

$$S \approx e^{-4\pi^2\sigma_w^2} \approx 0.2$$

or using the formula suggested in the original question:

$$S \approx e^{-\sigma_w^2} \approx 1 - \sigma_w^2 = 0.96$$

(b) This part required only a qualitative (not quantitative) assesment of the properties. In the following we added a bit more context, but in principle a short answer is sufficient as answer.

Spherical aberration reduces when $D \longrightarrow D' = D/2$, as spherical aberration corresponds to a (OPD) wavefront error $w_{SA} \propto r^4$ where r is the distance from the optical axis at the pupil aperture (the lens). If we don't balance the wavefront using defocus then w_{SA} is maximal for the largest $r = D/2$. Therefore:

$$D' \longrightarrow D/2 \quad \Rightarrow \quad \max w'_{SA} \longrightarrow \max \frac{w_{SA}}{2^4} = \frac{1}{16} \max w_{SA}$$

After balancing by applying defocus this scale can change but qualitatively the spherical aberration w_{SA} decreases. The transverse (TSA) and longitudinal spherical aberration (LSA) which can be derived from w_{SA} also decrease.

The root mean square optical path difference (RMS OPD) or wavefront error includes other aberrations beyond spherical aberration, but all of them would disappear for an infinitely small aperture, therefore **RMS OPD decreases** when reducing the clear diameter (better paraxial approximation).

The **Airy disk diameter will increase** as it scales inversely with the clear diameter D as can be seen from:

$$d' = 2.44\frac{\lambda f}{D'} = 2.44\frac{\lambda f}{D/2} = 2d$$

The Strehl ratio is maximally 1 and can be approximated by $S \approx e^{-4\pi^2\sigma_w^2}$, since the RMS OPD σ_w decreases when reducing D as stated above, **the Strehl ratio will increase**, meaning that the quality improves.

Question 2: ABCD propagation

Consider a thick plano-convex lens with surface curvature radii $R_1 = \infty$ and $R_2 = -20\,\mu\text{m}$. The lens has a clear diameter of $-20\,\mu\text{m}$ and thickness of $d = 10\,\mu\text{m}$.

(a) Use the transfer matrix method (ABCD matrices) to calculate an expression for the focal length of the lens.
(b) Derive how the focal length of the lens scales with the thickness d of the lens. Sketch this.

Solution (Q2)

(a) To derive the focal length we assume that incident rays are parallel to the optical axis ($\theta_0 = 0$) independent of their height y_0. We assume 4 matrices:

- M_1: Transfer matrix for planar surface 1 ($R_1 = \infty$),
- M_2: Propagation between surface 1 and surface 2 (thickness d),
- M_3: Transfer matrix for surface 2 ($R_2 = -20\,\mu\text{m}$),
- M_4: Propagation between surface 2 and the image plane (distance t).

The total ABCD matrix is then given by their product:

$$
\begin{aligned}
M = M_4 M_3 M_2 M_1 &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \\
&= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{n} \\ \frac{n-1}{R_2} & \frac{n-1}{n}\frac{d}{R_2} + 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \begin{pmatrix} y_4 = 0 \\ \theta_4 \end{pmatrix} &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{n} \\ \frac{n-1}{R_2} & \frac{n-1}{n}\frac{d}{R_2} + 1 \end{pmatrix} \begin{pmatrix} y_0 \\ \theta_0 = 0 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} 0 \\ \theta_4 \end{pmatrix} &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0 \frac{n-1}{R_2} \end{pmatrix} \\
\Rightarrow \quad t &= -\frac{R_2}{n-1} = \frac{20\,\mu\text{m}}{3/2 - 1} = 40\,\mu\text{m}
\end{aligned}
$$

This results in: $t = 40\,\mu\text{m}$, therefore the distance from surface 2 is $40\,\mu\text{m}$ independent of the initial height y_0. The length that we obtained here is the back focal length (BFL). However, the effective focal length (EFL) also called the focal length f, is the same as the BFL in this case: $f = \text{BFL} = t = 40\,\mu\text{m}$, because for a single thick lens they are related by:

$$
\text{BFL} = f\left(1 - \frac{d(n-1)}{nR_1}\right)
$$

and $R_1 = \infty$ when the first surface is planar.

(b) The back focal plane shifts with the thickness d but the back focal length BFL and thus the effective focal length f stay the same as they are independent of the thickness d (see the relation in (a)).

Question 3: Laser beam propagation

Consider a laser beam with wavelength $\lambda = 1000$ nm, beam width $w_0 = 2$ micron. Put a thin lens at distance 100 micron from the beam waist position to collimate the beam over a longer distance.

(a) What is the focal length of the thin lens? What is the beam width: $w'_0 = w'$ after collimation? Sketch the beam.
(b) After being collimated the beam encounters a circular detector with diameter 10 micron. Calculate the power ratio detected.

Solution (Q3)

(a) For the beam to be collimated, the beam waist position $z'_b = z_b + 100 = 100$ micron, is immediately after the lens. Perfect collimation is only possible for $w'_0 = \infty$, however, therefore we expect z'_0 to be finite. Before calculating the focal length f we derive the beam radius w'_0.

To find the beam radius w'_0 we look at the radius at the lens $w'_0 = w(z'_b)$:

$$w(z) = w_0\sqrt{1+\frac{z^2}{z_0^2}} = 2\sqrt{1+100^2/16\pi^2}\,\mu\text{m} \approx 100/2\pi \approx 16\,\mu\text{m}, \quad \text{with} \quad z_0 = \frac{\pi w_0^2}{\lambda} = 4\pi\,\mu\text{m}$$

The beam width just after the lens $w'_0 = w(z'_b) \approx 16\,\mu$m. The coherence length for the collimated beam is:

$$z'_0 = \frac{\pi w_0'^2}{\lambda} = \frac{\pi 4(1+100^2/16\pi^2)}{\lambda} \approx \frac{10^4}{4\pi} \approx 0.8\,\text{mm}$$

Now we can derive the focal length f of the thin lens: f is different from the distance for collimation, and given by the thin lens formula for Gaussian beams for the raddii w_0 and w'_0 as finite "objects". The relation between them is then:

$$w'_0 = \sqrt{1+{z'_0}^2/f^2}\,w_0 \quad \Rightarrow \quad f = \frac{w_0 z'_0}{\sqrt{{w'_0}^2 - w_0^2}} \approx 101.58\,\text{micron}$$

where z'_0 is the Rayleigh range for the collimated beam, and f the lens. Since $z'_0 = \frac{\pi {w'_0}^2}{\lambda} \gg f$ the focal length $f \approx z'_b = 100$ micron.

(b) The circular detector is assumed to be in the middle of the beam. The total power $P = \frac{1}{2}I_0\pi w_0^2$ is unknown because we don't know the intensity I_0. The power ratio detected depends on the radius of the detector $R_d = 5$ micron and can be calculated however, and is given by:

$$\frac{1}{P}\int_0^{R_d} I(r,z)2\pi r\,dr = 1 - e^{2R_d^2/w(z)^2} = 1 - e^{-2R_d^2/w'^2} \approx 1 - e^{-2\cdot 5^2/16^2} \approx 0.18$$

Hence the power detected is roughly 18 % of the power of the beam.