PHOT 445: Introduction to Quantum Optics

Final exam questions & solutions

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Grading of the course

The project will count for 50% of your grade. For the project points are given on the combined effort of the project and the oral explanation of it during the final exam.

The final exam comprises 50% of your grade and consists of 8 open questions. Each question is weighed equally for your grade. You will first answer the exam questions on paper and afterwards you can clarify further your written answers individually during the oral part. Clarification of the written answer during the oral part can influence your earned points for that question. For example, if a written answer is wrong but is afterwards corrected during the oral part, partial marks are given to the question.

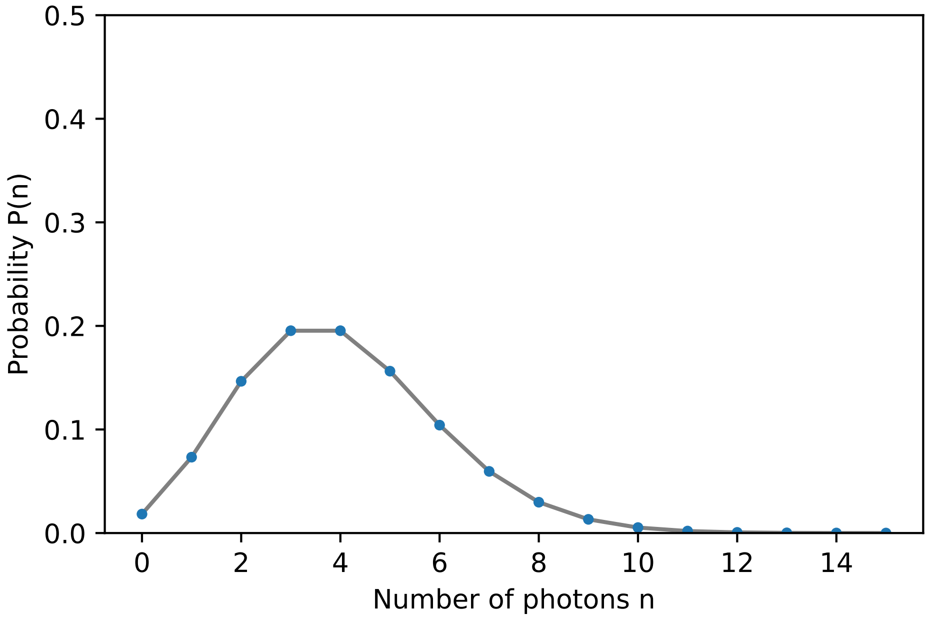
Exam questions

The different topics of the exam questions cover:

1. Photon statistics
2. Coherent, Bunched, and anti-bunched light
3. Squeezed states
4. Number states
5. Semi-classical: two-level atom and Rabi-oscillations
6. Quantized light and atoms: Jaynes-Cummings model
7. Quantum cryptography
8. Quantum logical gates and circuits

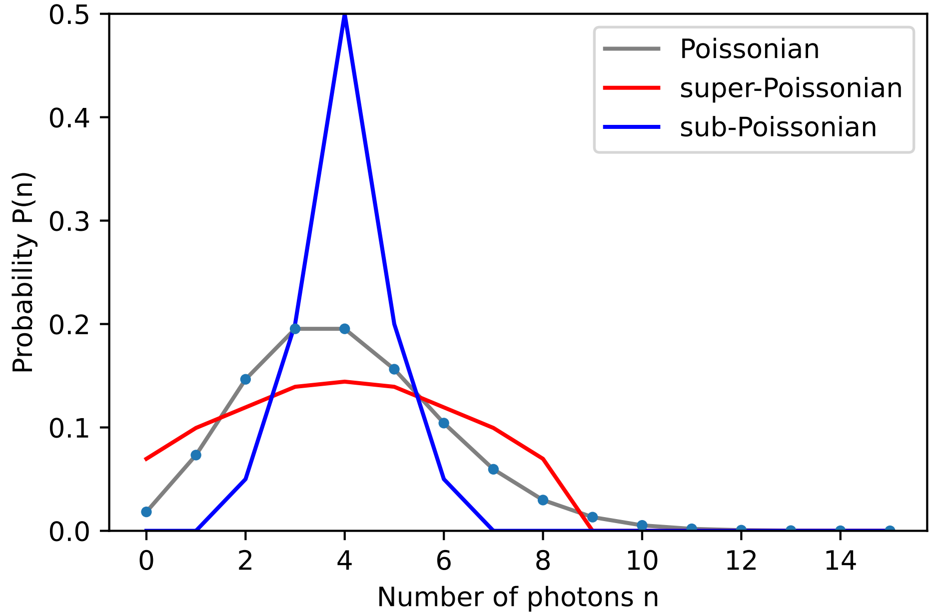
Question 1: Photon statistics

The below plot shows a Poissonian distribution with average photon number $\langle n \rangle = \bar{n} = 4$ and standard deviation $\Delta n = \sqrt{\bar{n}} = 2$. Add two other distributions with the same average photon number $\bar{n} = 4$: one sub-Poissonian and one super-Poissonian distribution.



Answer:

For a super-Poissonian distribution the standard deviation is larger than for coherent light: $\Delta n > \sqrt{\bar{n}} = 2$ ("broader" distribution) while keeping $\bar{n} = 4$. For the sub-Poissonian distribution we require $\Delta n < \sqrt{\bar{n}} = 2$ ("narrower" distribution), again with $\bar{n} = 4$. We sketched a possible distribution for both cases below:



Question 2: Coherent, Bunched, and anti-bunched light

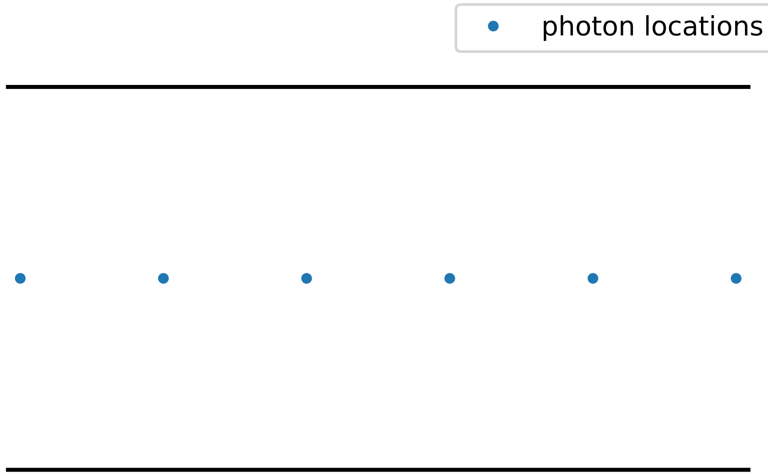
The second order correlation function $g^{(2)}(\tau)$ given by:

$$g^{(2)}(\tau) = \frac{\langle I(t)\, I(t+\tau)\rangle}{\langle I(t)\rangle\, \langle I(t+\tau)\rangle}$$

A source that emits single photons can be used to create a beam of anti-bunched light. Do you expect the value of $g^{(2)}(\tau \approx 0)$ to be larger or smaller than 1? Can you sketch the photon positions for such anti-bunched light in a 1D light beam?

Answer:

For anti-bunched light $g^{(2)}(\tau \approx 0) < 1$. The photons are more regularly spaced than in the coherent case. As an example of anti-bunched light we can take photons being send out periodically, i.e. perfect regular distribution in space. For a 1D beam we sketched periodically spaced photon positions below (dots represent photon locations):



Question 3: Squeezed states

Coherent light can be expressed in the phase space of the dimensionless quadratures X_1 and X_2

$$\begin{cases} X_1(t) = \sqrt{\dfrac{\epsilon_0 V}{4\hbar\omega}}\mathcal{E}_0 \sin(\omega t) \\ X_2(t) = \sqrt{\dfrac{\epsilon_0 V}{4\hbar\omega}}\mathcal{E}_0 \cos(\omega t) \end{cases}$$

by the coherent state $|\alpha\rangle$ with complex number $\alpha = |\alpha|e^{i\theta} = X_1 + \mathrm{i}\,X_2$. The uncertainty relation in X_1 and X_2 is given by

$$\Delta X_1\,\Delta X_2 \geq \frac{1}{4}$$

A corresponding uncertainty relation in photon number and phase can be found:

$$\Delta n\,\Delta\theta \geq \frac{1}{2}$$

Based on the above uncertainty relation (and the schematic figure) can you answer the following questions:

- What do you expect from the phase uncertainty $\Delta\theta$ when $|\alpha|$ increases?
- What is the phase uncertainty $\Delta\theta$ for the vacuum state?

Answer:

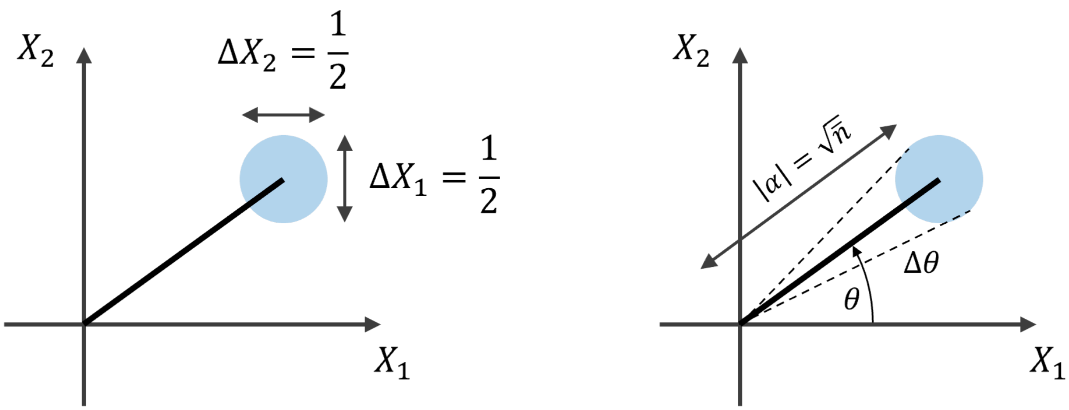


Figure 1: Coherent state $|\alpha\rangle$

- What do you expect from the phase uncertainty $\Delta\theta$ when $|\alpha|$ increases?

The uncertainty on the phase $\Delta\theta$ decreases if the uncertainty in number of photons Δn increases. For a coherent state $|\alpha| = \Delta n$ therefore, if $|\alpha|$ increases, the uncertainty on the phase should decrease.

- What is the phase uncertainty $\Delta\theta$ for the vacuum state?

For the vacuum state the phase becomes completely uncertain (as $\Delta n = |\alpha| = 0 \Rightarrow \Delta\theta \to \infty$). This can also geometrically be seen as the angle $\Delta\theta$ increases when the uncertainty circle moves closer to the origin (the uncertainty area is minimal and thus constant for a coherent state as $\Delta X_1 = \Delta X_2 = \frac{1}{2}$, assuming the isotropic case).

Question 4: Number states

A coherent state with value α is described by:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Prove that the expectation value (average) of the number of photons of a coherent state $|\alpha\rangle$ is $\langle\alpha| \hat{n} |\alpha\rangle = \bar{n} = |\alpha|^2$.

Use hereby that the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ and applying the annihilation operator $\hat{a}$ on a coherent state results in $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$.

Answer:

To prove that $\langle\alpha| \hat{n} |\alpha\rangle = |\alpha|^2$ we first use $\hat{n} = \hat{a}^\dagger \hat{a}$ and calculate then $\langle\alpha| \hat{a}^\dagger\hat{a} |\alpha\rangle$:

$$
\begin{aligned}
\langle\alpha|\,\hat{n}\,|\alpha\rangle &= \langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle \\
&= \Big(\langle\alpha|\,\hat{a}^\dagger\Big)\Big(\hat{a}\,|\alpha\rangle\Big) \\
&= \Big(\langle\alpha|\,\alpha^*\Big)\Big(\alpha\,|\alpha\rangle\Big) \qquad \text{because } \langle\alpha|\hat{a}^\dagger = \Big(\hat{a}\,|\alpha\rangle\Big)^\dagger = \Big(\alpha\,|\alpha\rangle\Big)^\dagger = \langle\alpha|\alpha^* \\
&= \langle\alpha|\,\alpha^*\,\alpha\,|\alpha\rangle \\
&= \langle\alpha|\,|\alpha|^2\,|\alpha\rangle \\
&= |\alpha|^2\,\langle\alpha|\alpha\rangle \\
&= |\alpha|^2
\end{aligned}
$$

Question 5: Semi-classical - two-level atom and Rabi-oscillations

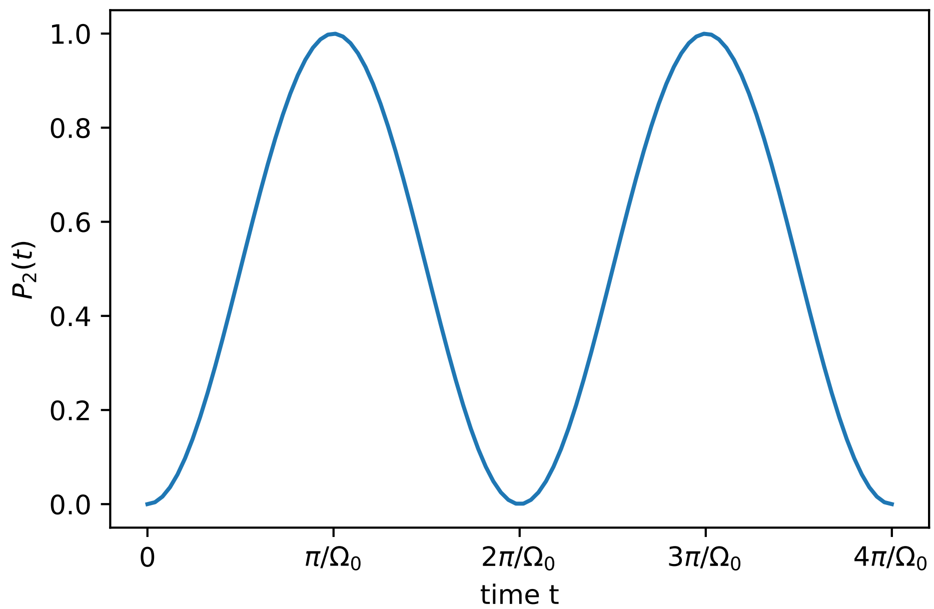
In a two-level atom interacting with a single mode of light we can observe Rabi oscillations where the two-level atom is oscillating between the ground and its excited state. In case the excitation energy $\hbar\omega_{21}$ of the two-level atom equals the frequency of the light $\hbar\omega$ then the detuning is zero ($\delta = \omega - \omega_{21} = 0$) and the probability to be in the excited state P_2 is given by

$$P_2(t) = \sin^2(\Omega_0\, t/2)$$

Can you make a sketch of the probability P_2 as function of the time t? What is the probability P_1 for being in the ground state?

Answer:

Below is the sketch of the probability $P_2(t)$ to be in the excited state. The probability to be in the ground state is $P_1(t) = 1 - P_2(t) = \cos^2(\Omega_0\, t/2)$



Question 6: Jaynes-Cummings model

The Jaynes-Cummings model takes into account a single two-level atom and quantized light of a single mode. The Hamiltonian is written as:

$$\widehat{H}_{JC} = \widehat{H}_a + \widehat{H}_c + \widehat{H}_I$$

Where the parts are the Hamiltonian of the free atom, the cavity, and the interaction between them. In the rotating wave approximation, the Hamiltonian becomes:

$$\widehat{H}_{JC} = \hbar\omega_a\sigma^\dagger\sigma + \hbar\omega_c a^\dagger a + \hbar g(\sigma^\dagger a + \sigma a^\dagger)$$

where the first term is sometimes rewritten as $\frac{1}{2}\hbar\omega_a\sigma_z$

- Can you explain the different terms?
- What does g represent? (in words)

Answer:

The different terms of the Hamiltonian:

- The first term $\hbar\omega_a\widehat{\sigma}^\dagger\widehat{\sigma}$: Represents the "number" operator for the excited state, where $\widehat{\sigma}$ brings the excitation back to the ground state (annihilates an "excitation"). There are only two energy levels and $\hbar\omega_a$ is the excitation energy between them.

- The second term $\hbar\omega_c \hat{a}^\dagger \hat{a}$ is similar to the first term but for "free space" photons in the cavity mode. $\hat{a}$ is the annihilation operator for photons.
- The third term $\hbar g(\hat{\sigma}^\dagger \hat{a} + \hat{\sigma}\hat{a}^\dagger)$ represents the interaction between the atom and the photons in the cavity: here $\hat{\sigma}^\dagger \hat{a}$ represents the absorption of a photon bringing the atom in the excited state. $\hat{\sigma}\hat{a}^\dagger$ represents the spontaneous emission with the atom going from the excited state to the ground state.

The coupling constant g incorporates the dipole moment matrix elements $\vec{d}_{12}$, polarization vector $\vec{\epsilon}_{\vec{k}}$, and electric field within the cavity. These are constants depending on cavity and atom. The expression for g is given by (not required for this question):

$$g = g_{\vec{k}} = \sqrt{\frac{\omega_{\vec{k}}}{\mathcal{E}_0 V}}\, \vec{d}_{12} \cdot \vec{\epsilon}_{\vec{k}}$$

where $\vec{k}$ is the wave vector of the cavity mode, $\omega_{\vec{k}}$ its frequency and V the cavity volume).

Question 7: Quantum cryptography

When using linearly polarized light to compose a signal, how can you represent bits of information?

Answer:

A bit of information can be represented by polarization under different angles, the (linear) polarization can written in the basis of for example vertically and horizontally polarized light. Since they form an orthonormal basis we can use e.g. vertically polarized light as $|0\rangle$ or 0-bit and horizontally polarized light as $|1\rangle$ or 1-bit.

Question 8: Quantum logical gates and circuits

A qubit q can be written as $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$ or in column vector notation: $q = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$.

The Hadamard gate (H) and the NOT gate (X) are single qubit logical gates given by the following matrix representations:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If you start from a qubit q in state $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and you apply first a Hadamard gate and afterwards a NOT gate, what is the resulting qubit state q'.

$$q' = X \cdot H \cdot q$$

Answer:

The resulting qubit state q' is given by:

$$\begin{aligned}
q' &= X \cdot H \cdot q \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} (-|0\rangle + |1\rangle)
\end{aligned}$$