PHOT 445: Introduction to Quantum Optics

Final exam questions

Michaël Barbier, Spring semester (2023-2024)

Grading of the course

The project will count for 50% of your grade. For the project points are given on the combined effort of the project and the oral explanation of it during the final exam.

The final exam comprises 50% of your grade and consists of 8 open questions. You will first answer the exam questions on paper and afterwards you can clarify further your written answers individually during the oral part.

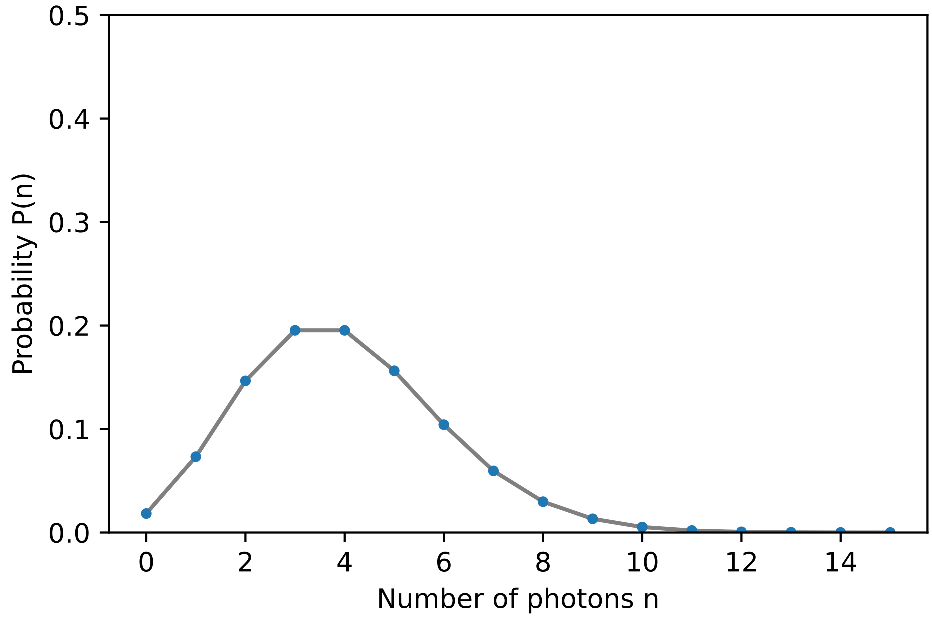
Example exam questions

The different topics of the exam questions cover:

1. Photon statistics
2. Coherent, Bunched, and anti-bunched light
3. Squeezed states
4. Number states
5. Semi-classical: two-level atom and Rabi-oscillations
6. Quantized light and atoms: Jaynes-Cummings model
7. Quantum cryptography
8. Quantum logical gates and circuits

Question 1: Photon statistics

The below plot shows a Poissonian distribution with average photon number $\langle n \rangle = \bar{n} = 4$ and standard deviation $\Delta n = \sqrt{\bar{n}} = 2$. Add two other distributions with the same average photon number $\bar{n} = 4$: one sub-Poissonian and one super-Poissonian distribution.



Question 2: Coherent, Bunched, and anti-bunched light

The second order correlation function $g^{(2)}(\tau)$ given by:

$$g^{(2)}(\tau) = \frac{\langle I(t)\, I(t+\tau)\rangle}{\langle I(t)\rangle \, \langle I(t+\tau)\rangle}$$

A source that emits single photons can be used to create a beam of anti-bunched light. Do you expect the value of $g^{(2)}(\tau \approx 0)$ to be larger or smaller than 1? Can you sketch the photon positions for such anti-bunched light in a 1D light beam?

Question 3: Squeezed states

Coherent light can be expressed in the phase space of the dimensionless quadratures X_1 and X_2

$$\begin{cases} X_1(t) = \sqrt{\dfrac{\epsilon_0 V}{4\hbar\omega}}\mathcal{E}_0 \sin(\omega t) \\ X_2(t) = \sqrt{\dfrac{\epsilon_0 V}{4\hbar\omega}}\mathcal{E}_0 \cos(\omega t) \end{cases}$$

by the coherent state $|\alpha\rangle$ with complex number $\alpha = |\alpha|e^{i\theta} = X_1 + \mathrm{i}\, X_2$. The uncertainty relation in X_1 and X_2 is given by

$$\Delta X_1\, \Delta X_2 \,\geq\, \frac{1}{4}$$

A corresponding uncertainty relation in photon number and phase can be found:

$$\Delta n\, \Delta\theta \,\geq\, \frac{1}{2}$$

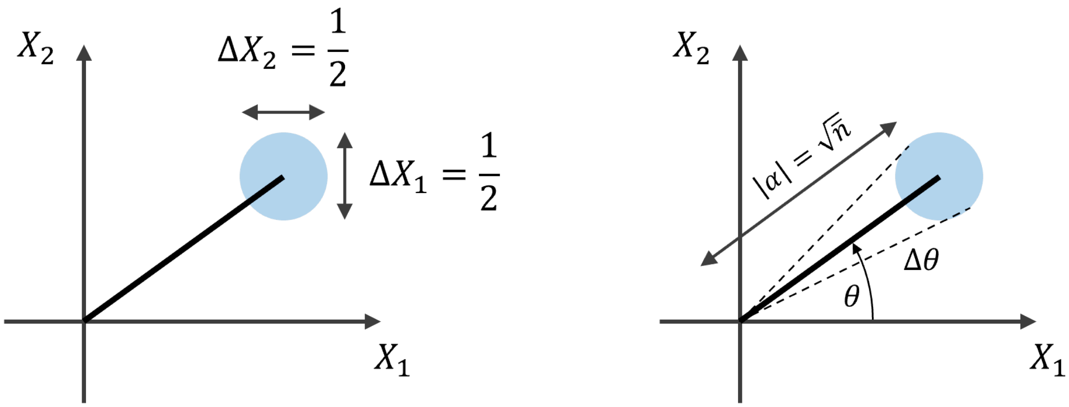


Figure 1: Coherent state $|\alpha\rangle$

Based on the above uncertainty relation (and the schematic figure) can you answer the following questions:

- What do you expect from the phase uncertainty $\Delta\theta$ when $|\alpha|$ increases?
- What is the phase uncertainty $\Delta\theta$ for the vacuum state?

Question 4: Number states

A coherent state with value α is described by:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle$$

Prove that the expectation value (average) of the number of photons of a coherent state $|\alpha\rangle$ is $\langle\alpha|\,\hat{n}\,|\alpha\rangle = \bar{n} = |\alpha|^2$.

Use hereby that the number operator $\hat{n} = \hat{a}^\dagger\, \hat{a}$ and applying the annihilation operator $\hat{a}$ on a coherent state results in $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$.

Question 5: Semi-classical - two-level atom and Rabi-oscillations

In a two-level atom interacting with a single mode of light we can observe Rabi oscillations where the two-level atom is oscillating between the ground and its excited state. In case the excitation energy $\hbar\omega_{21}$ of the two-level atom equals the frequency of the light $\hbar\omega$ then the detuning is zero ($\delta = \omega - \omega_{21} = 0$) and the probability to be in the excited state P_2 is given by

$$P_2(t) = \sin^2(\Omega_0\, t/2)$$

Can you make a sketch of the probability P_2 as function of the time t? What is the probability P_1 for being in the ground state?

Question 6: Jaynes-Cummings model

The Jaynes-Cummings model takes into account a single two-level atom and quantized light of a single mode. The Hamiltonian is written as:

$$\hat{H}_{JC} = \hat{H}_a + \hat{H}_c + \hat{H}_I$$

Where the parts are the Hamiltonian of the free atom, the cavity, and the interaction between them. In the rotating wave approximation, the Hamiltonian becomes:

$$\hat{H}_{JC} = \hbar\omega_a \sigma^\dagger \sigma + \hbar\omega_c a^\dagger a + \hbar g(\sigma^\dagger a + \sigma a^\dagger)$$

where the first term is sometimes rewritten as $\frac{1}{2}\hbar\omega_a \sigma_z$

- Can you explain the different terms?
- What does g represent? (in words)

Question 7: Quantum cryptography

When using linearly polarized light to compose a signal, how can you represent bits of information?

Question 8: Quantum logical gates and circuits

A qubit q can be written as $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$ or in column vector notation: $q = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$.

The Hadamard gate (H) and the NOT gate (X) are single qubit logical gates given by the following matrix representations:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If you start from a qubit q in state $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and you apply first a Hadamard gate and afterwards a NOT gate, what is the resulting qubit state q'.

$$q' = X \cdot H \cdot q$$