PHOT 301: Quantum Photonics

LECTURE 12

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DIRAC NOTATION

BRACKETS: BRA'S AND KETS

- Inner product in matrix notation (separate "vectors")

$$\langle\alpha|\beta\rangle = \begin{pmatrix} a_1^* & a_2^* & \dots & a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + \dots a_n^* b_n$$

- "bra" acts on the ket by row vector multiplication
- "bra" *vector* is separate from the "ket" vector: bra sits in a **dual vector space**
- Now with possible infinite basis:

$$\langle\alpha| = \sum_j a_j^* (\dots)_j \quad \longrightarrow \quad \langle\alpha| = \int \alpha^* (\dots) dx$$

BRACKETS: BRA'S AND KETS

- Kets are vectors in vector space
- Bra's are vectors in dual space
- In finite dimensions:
  - kets are column vectors
  - bra's are complex conjugate row vectors

$$\langle \mathrm{bra}| = \langle \alpha| = \begin{pmatrix} a_1^* & a_2^* & \dots & a_n^* \end{pmatrix}$$

$$|\mathrm{ket}\rangle = |\beta\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

DUAL SPACE AND HERMITIAN CONJUGATES

- Converting a $|\mathrm{ket}\rangle$ to a $\langle bra|$ and vice versa:

$$\langle\alpha| = |\alpha\rangle^\dagger$$

- An operator acting on a $\langle bra|$:

$$\langle\alpha|\hat{Q}^\dagger = \langle\hat{Q}\alpha| = \left(\hat{Q}|\alpha\rangle\right)^\dagger$$

$\longrightarrow$ operators can *act to the left* as this is allowed by associativity

- *Why is this?* See definition of Hermitian conjugate of operators:

$$\langle\hat{Q}^\dagger\alpha|\beta\rangle = \langle\alpha|\hat{Q}\beta\rangle$$

IN FINITE DIMENSIONS: MATRIX-FORMALISM

- Example in two dimensions, an operator acting on a $|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$:

$$\hat{Q}|\alpha\rangle = Q\mathbf{a} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Q_{11}a_1 + Q_{12}a_2 \\ Q_{21}a_1 + Q_{22}a_2 \end{pmatrix}$$

The Hermitian conjugate gives

$$\langle\alpha|\hat{Q}^\dagger = \mathbf{a}^\dagger Q^\dagger = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \begin{pmatrix} Q_{11}^* & Q_{21}^* \\ Q_{12}^* & Q_{22}^* \end{pmatrix} = \begin{pmatrix} Q_{11}^* a_1^* + Q_{12}^* a_2^* & Q_{21}^* a_1^* + Q_{22}^* a_2 \end{pmatrix}$$

For this example we indeed see that:

$$\langle\alpha|\hat{Q}^\dagger = \left(\hat{Q}|\alpha\rangle\right)^\dagger$$

THE PROJECTION OPERATOR

- The projection operator defined for a normalized $|\alpha\rangle$:

$$\hat{P}_\alpha = |\alpha\rangle\langle\alpha|$$

$\longrightarrow$ Projects any other vector $|\beta\rangle$ onto the direction of $|\alpha\rangle$:

$$\hat{P}_\alpha|\beta\rangle = (\langle\alpha|\beta\rangle)\,|\alpha\rangle$$

Example: projection in two dimensions

$$|\alpha\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}, \quad |\beta\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\hat{P}_\alpha|\beta\rangle = |\alpha\rangle\langle\alpha|\beta\rangle = \frac{1}{5}\begin{pmatrix} 1 \\ 2i \end{pmatrix}\begin{pmatrix} 1 & -2i \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{2}{5}(1-i)\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

THE PROJECTION OPERATOR: EXAMPLE

Example: projection in two dimensions

$$|\alpha\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}, \quad |\beta\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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The operator itself is an **outer product**:

$$\hat{P}_\alpha = |\alpha\rangle\langle\alpha| = \frac{1}{5}\begin{pmatrix} 1 \\ 2i \end{pmatrix}\begin{pmatrix} 1 & -2i \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & -2i \\ 2i & 4 \end{pmatrix}$$

Two-dimensional vector spaces are actually useful: Spin, the two-level atom approximation, etc.

IDENTITY OPERATORS

- If we have a complete basis $\{\,|e_n\rangle\}$
- Projection operator:

$$\hat{P}_n = |e_n\rangle\langle e_n|$$

Then the identity operator can be written as:

$$\boxed{\sum_n \hat{P}_n = \sum_n |e_n\rangle\langle e_n| = \hat{1}}$$

Or for a continuous spectrum and eigenfunction basis:

$$\langle e_z|e'_z\rangle = \delta(z-z') \qquad \boxed{\int |e_z\rangle\langle e_z|\,dz = \hat{1}}$$

FUNCTIONS OF OPERATORS: POWER SERIES

- Sums and products of operators, **order is important**:

$$(\hat{Q} + c\hat{R})|\alpha\rangle = \hat{Q}|\alpha\rangle + c\hat{R}|\alpha\rangle \qquad \hat{Q}\hat{R}|\alpha\rangle = \hat{Q}\left(\hat{R}|\alpha\rangle\right)$$

- Functions of operators are represented by their **power series**
- Likewise with matrices (also operators in our case):

$$e^{\hat{Q}} = 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \dots$$

$$\frac{1}{1-\hat{Q}} = 1 + \hat{Q} + \hat{Q}^2 + \hat{Q}^3 + \hat{Q}^4 + \dots$$

$$\ln(1+\hat{Q}) = \hat{Q} - \frac{1}{2}\hat{Q}^2 + \frac{1}{3}\hat{Q}^3 - \frac{1}{4}\hat{Q}^4 \dots$$

THE WAVE FUNCTION IN HILBERT SPACE

- The wave function of a quantum state $|\Psi(t)\rangle$

$$\Psi(x,t) = \langle x|\Psi(t)\rangle, \qquad \hat{x}|x\rangle = x_0|x\rangle$$

$\longrightarrow$ x_0 are eigenvalues of position operator $\hat{x}$

$$\langle x_0|\Psi(t)\rangle = \int_{-\infty}^{\infty} \delta(x - x0)\psi(x)dx = \psi(x_0)$$

MOMENTUM EIGENVECTORS?

Momentum eigenvalue equation:

$$\hat{p}|\Psi\rangle = p|\Psi\rangle$$

- Filling in momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$:

$$\frac{d\psi_p(x)}{dx} = \frac{ip}{\hbar}\psi_p(x)$$

This differential equation has solution:

$$\psi_p(x) = Ae^{ipx/\hbar} = \frac{1}{\sqrt{2\pi}}e^{ipx/\hbar}$$