PHOT 301: Quantum Photonics

LECTURE 11

Michaël Barbier, Fall semester (2024-2025)

SUMMARY OF WHAT WE KNOW

- States $|\Psi\rangle$ can be represented by the wave function:
  - $\Psi(x,t) = \langle x|\Psi(t)\rangle$
  - **this is similar to a vector in vector component notation**

- Observables are measurable quantities ("real" results)
- Observables Q correspond to operators $\hat{Q}$:
  - Linear operators $\longrightarrow$ $\hat{Q}\alpha\rangle$
  - Hermitian $\longrightarrow$ $\hat{Q}^\dagger = \hat{Q}$

- Observable operators have a spectrum of eigenvalues
- Spectrum: **discrete** $(q_n\,,\ |f_n\rangle)$, **continuous** $(q(z),\ |f_z\rangle)$, or a **mixture**

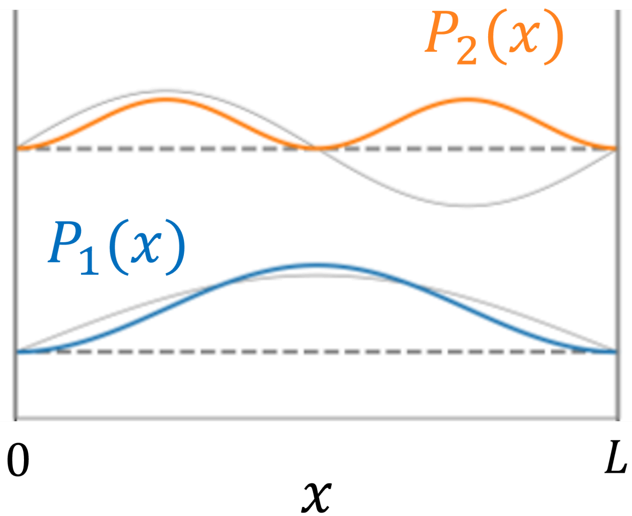
OBSERVABLES, OPERATORS AND COLLAPSE

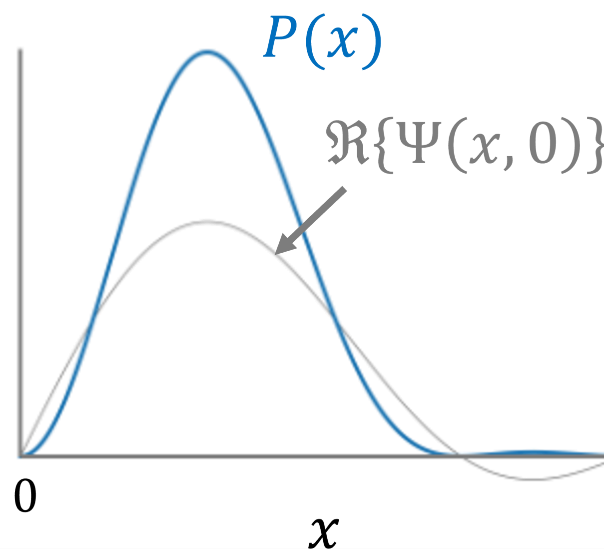
- We can measure observables:
  - position and momentum of a particle,
  - energy of a particle in a potential,
  - excitation-level of an electron in an atom
  - spin of an electron
  - …

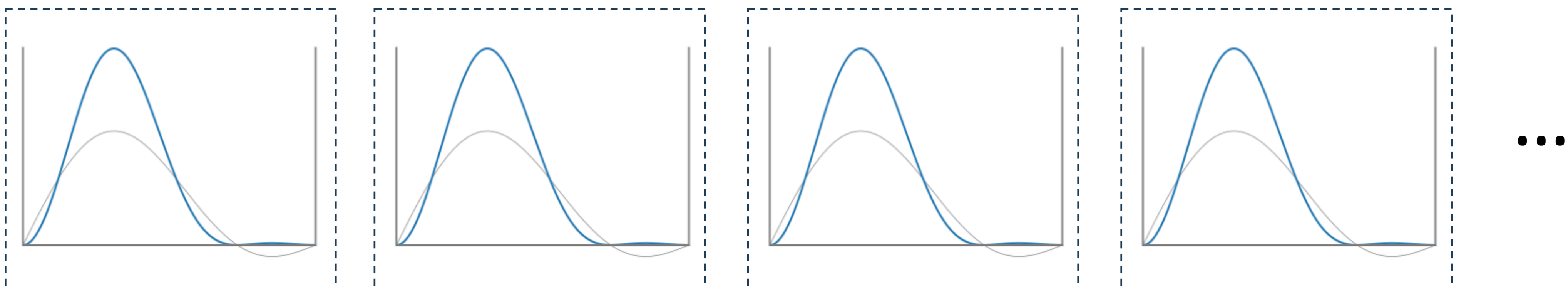
- Before measurement
  - superposition of eigenstates
  - Probability to find a particle in x: $|\Psi(x,t)|^2$
  - $\Psi(x,t) = \sum c_n(t)\psi_n(x) \longrightarrow P(n) = |c_n(t)|^2$

- Measurement: system collapses to single eigenstate

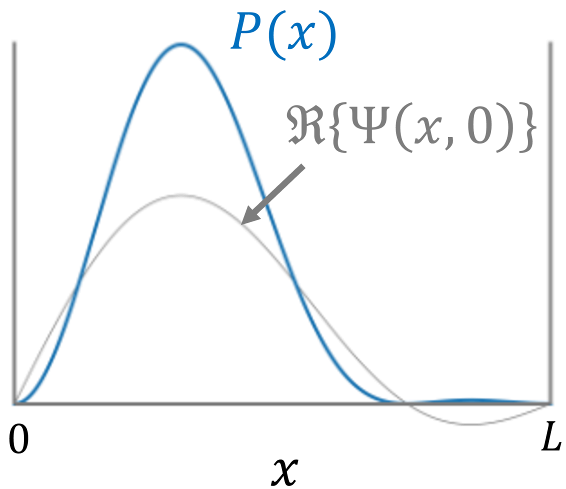
INFINITE WELL

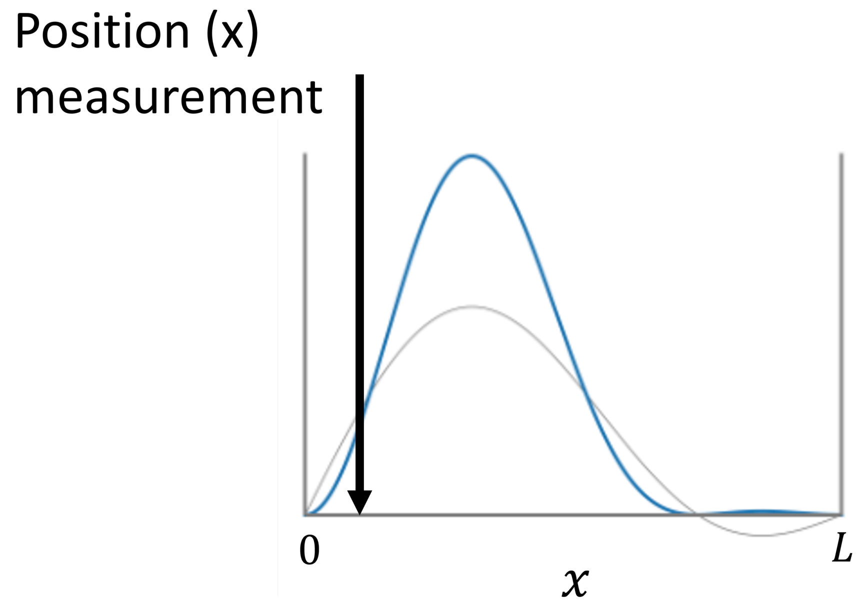




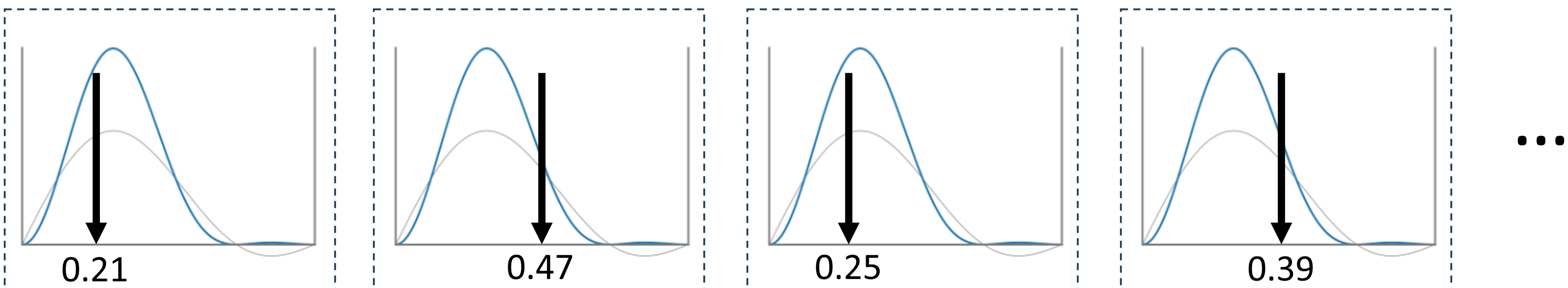


INFINITE WELL: OBSERVABLE POSITION

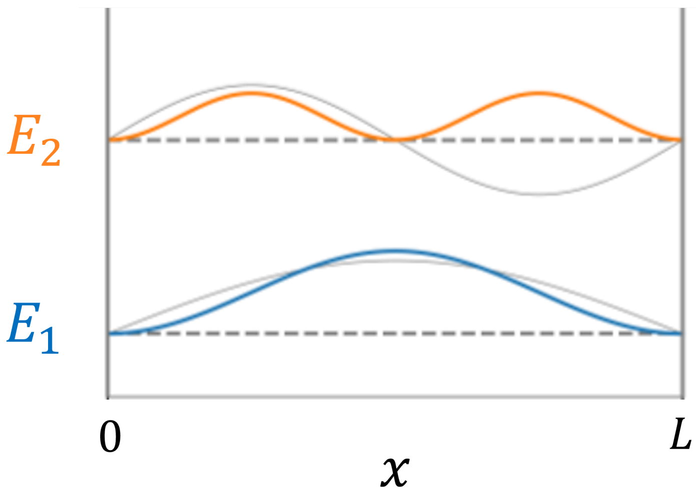




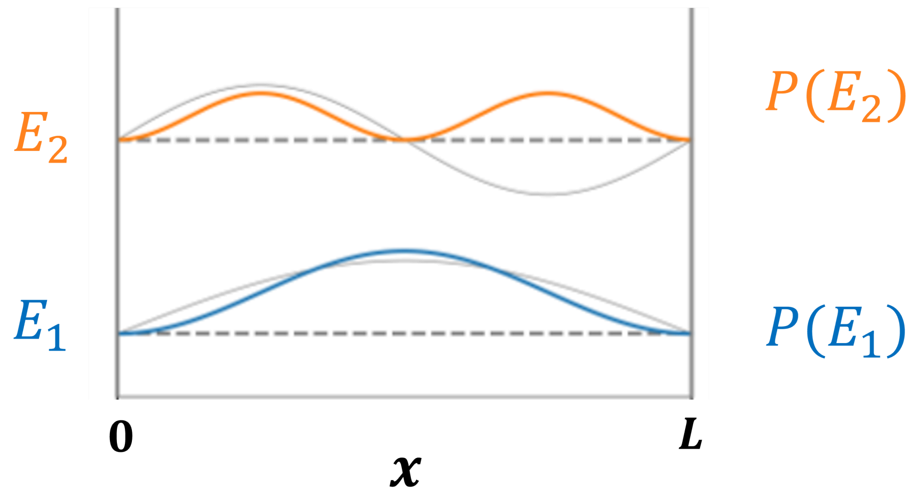
Every measurement probabilistic BUT average position $\langle x \rangle \propto \int_0^L P(x)\, x\, dx$

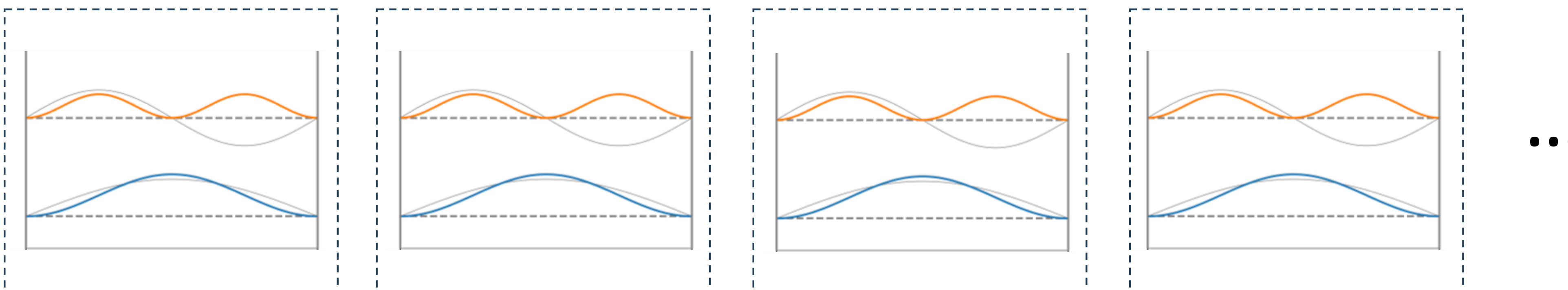


INFINITE WELL: ENERGIES

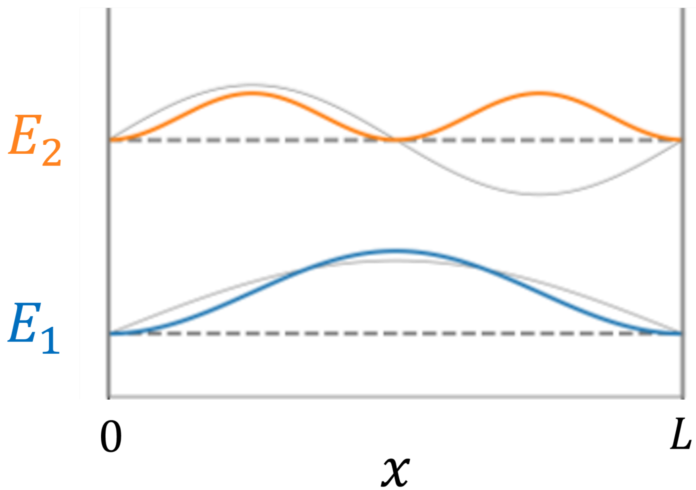


Energy (E_n) measurement





INFINITE WELL: OBSERVABLE ENERGY



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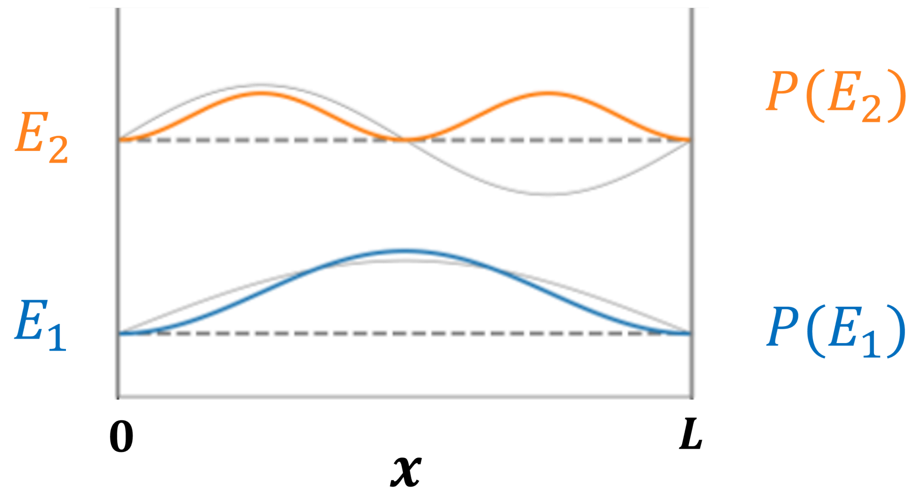
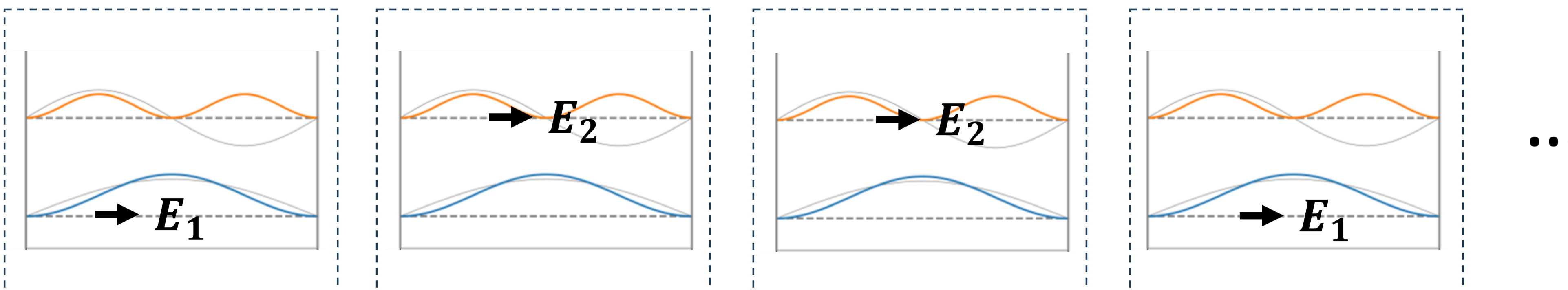
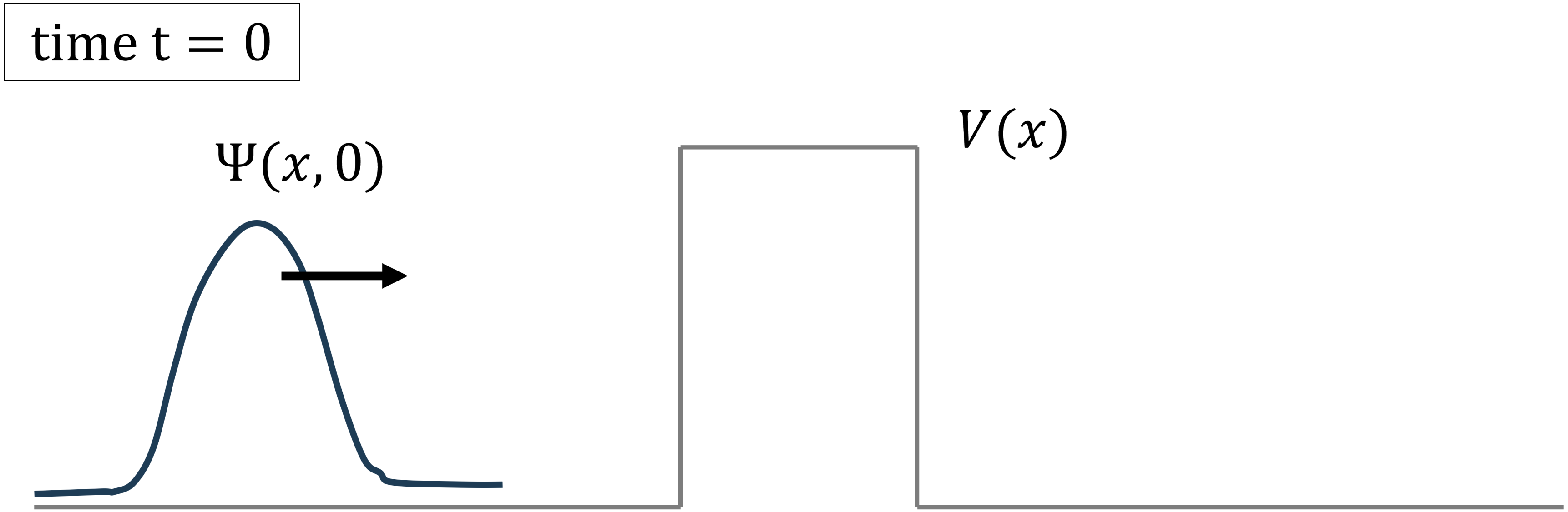
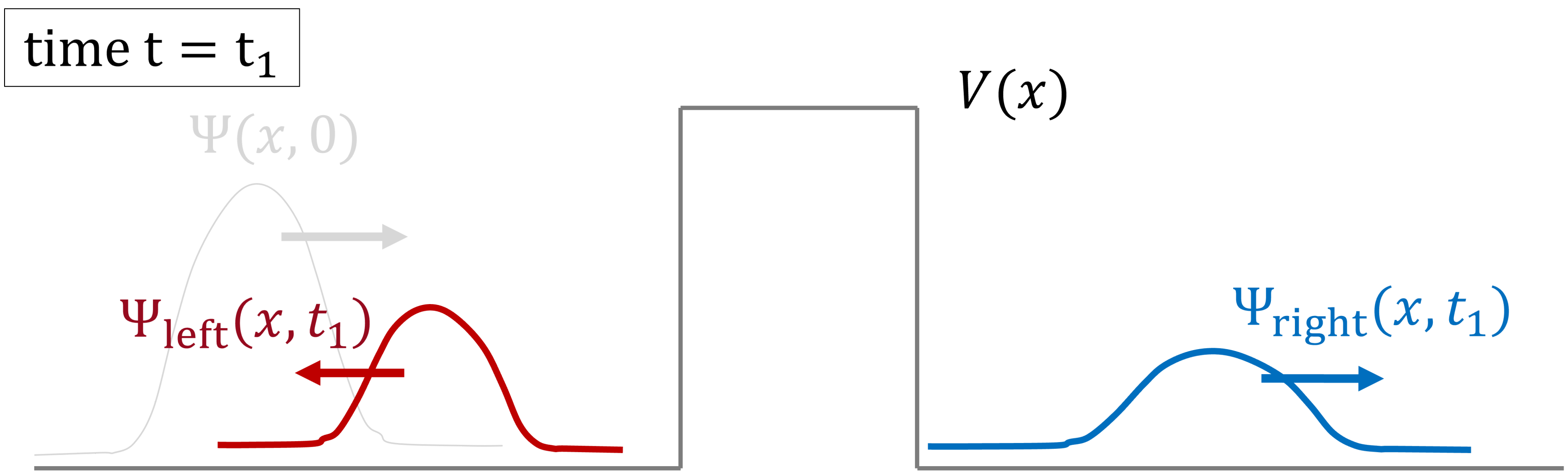


Every measurement probabilistic BUT average energy $\langle H \rangle \propto \sum_n P(E_n) E_n$

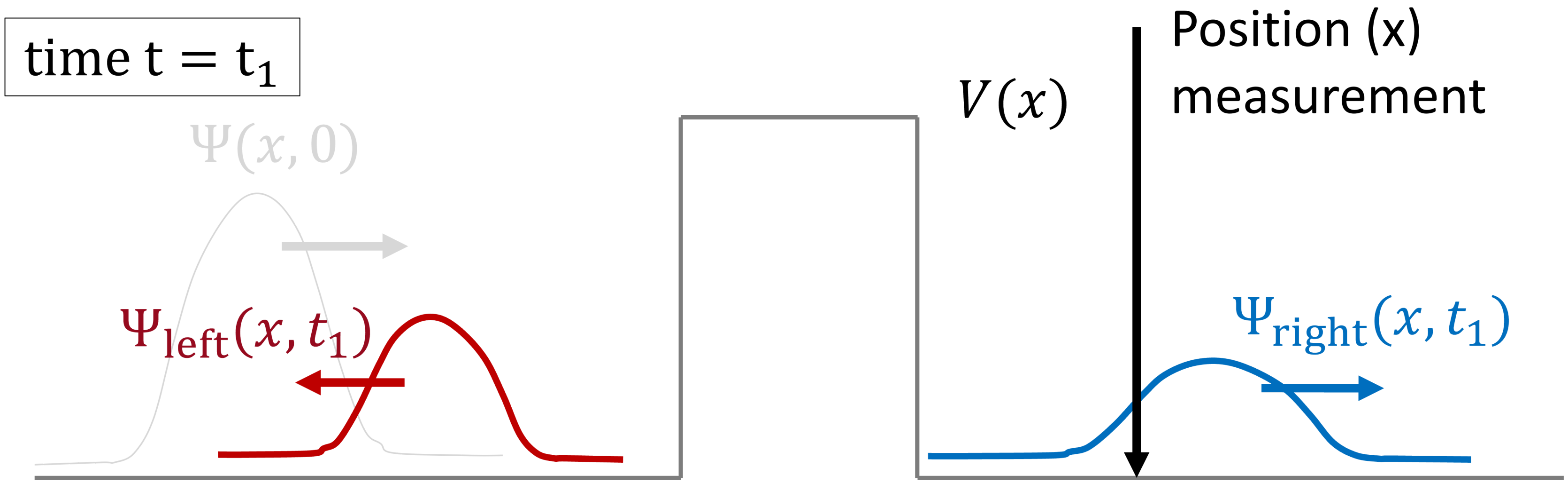


WAVEPACKET INCIDENT ON BARRIER





WAVEPACKET: OBSERVABLE POSITION



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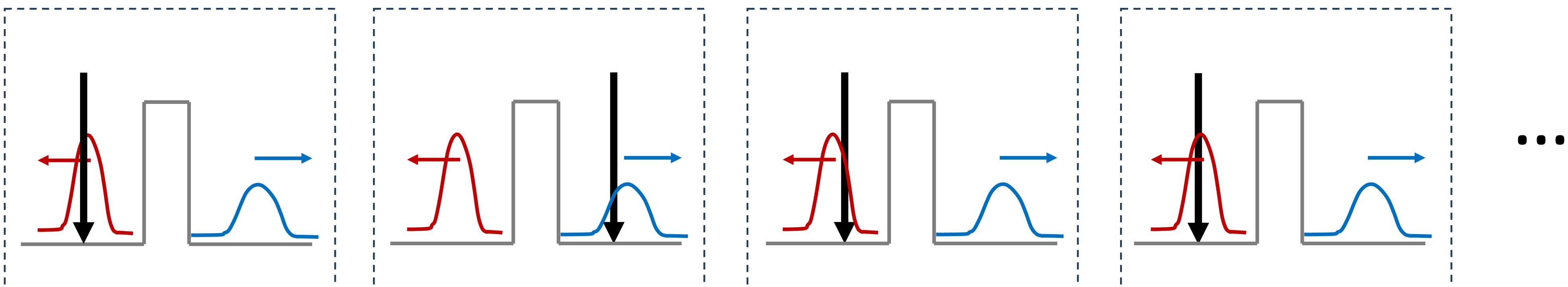

OBSERVABLES, OPERATORS AND COLLAPSE

- State of a quantum system: $|\Psi\rangle$
- Wave function represents state: $\langle x|\Psi(t)\rangle \longrightarrow \Psi(x,t)$
- Observable is something we can measure (a real number)
- Observable Q corresponds to an Hermitian operator $\hat{Q}$
- Measuring NOT same as applying operator $\hat{Q}|\Psi\rangle$
- Measurement operators DON'T always commute (**incompatible** observables)
- Incompatible observables $\longrightarrow$ **NO common basis** of eigenfunctions

UNCERTAINTY PRINCIPLE

- Heisenberg uncertainty principle

$$\sigma_x \sigma_p \leq \frac{\hbar}{2}$$

- Commutator is nonzero:

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

- Can’t measure position and momentum at the same time
- Measuring position *destroys* the momentum measurement

GENERALIZED UNCERTAINTY PRINCIPLE

- **General** uncertainty principle is related to the commutator

$$\boxed{\sigma_A^2 \sigma_B^2 \leq \left(\frac{1}{2i} \langle \left[\hat{A}, \hat{B} \right] \rangle \right)^2}$$

- Number between brackets is real but can be negative
- We need the square at the right-hand-side
- Commutating operators $\longrightarrow$ no restriction on σ_A, σ_B

How to proof this?

EXAMPLE UNCERTAINTY PRINCIPLE

- General uncertainty principle for position/momentum
- The commutator for $\hat{x}$ and $\hat{p}$:

$$[\hat{x}, \hat{p}] = i\hbar$$

Fill in in general uncertainty formula:

$$\sigma_A^2 \sigma_B^2 \leq \left(\frac{1}{2i} \langle \left[\hat{A}, \hat{B} \right] \rangle \right)^2$$

$$\Rightarrow \quad \sigma_x^2 \sigma_p^2 \leq \left(\frac{1}{2i} \langle [\hat{x}, \hat{p}] \rangle \right)^2 = \left(\frac{1}{2i} \langle i\hbar \rangle \right)^2 = \frac{\hbar^2}{4}$$

$$\Rightarrow \quad \sigma_x \sigma_p \leq \frac{\hbar}{2}$$

$\longrightarrow$ Heisenberg uncertainty principle

COMMUTATORS AND UNCERTAINTY

$$\boxed{\sigma_A^2 \sigma_B^2 \leq \left(\frac{1}{2i}\langle\left[\hat{A}, \hat{B}\right]\rangle\right)^2}$$

- **Compatible observables**: Commutating observables $[\hat{A}, \hat{B}] = 0$
  - Measurements independent, order doesn’t matter
  - No restriction on the common **uncertainty** of the measurement
  - A common basis of eigenstates can be found

- **Incompatible observables**: Commutating observables $[\hat{A}, \hat{B}] > 0$
  - **Order** of the measurement **matters** !
  - **Minimum uncertainty** on the measurements according to formula
  - **NO common basis** of eigenstates can be found

DIRAC NOTATION

VECTORS IN HILBERT SPACE

- The wave function of a quantum state $|\Psi(t)\rangle$

$$\Psi(x,t) = \langle x|\Psi(t)\rangle, \qquad \hat{x}|x\rangle = x_0|x\rangle$$

$\longrightarrow$ x_0 are eigenvalues of position operator $\hat{x}$

$$\langle x_0|\Psi(t)\rangle = \int_{-\infty}^{\infty} \delta(x - x0)\psi(x)dx = \psi(x_0)$$

MOMENTUM EIGENVECTORS?

Momentum eigenvalue equation:

$$\hat{p}|\Psi\rangle = p|\Psi\rangle$$

- Filling in momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$:

$$\frac{d\psi_p(x)}{dx} = \frac{ip}{\hbar}\psi_p(x)$$

This differential equation has solution:

$$\psi_p(x) = Ae^{ipx/\hbar} = \frac{1}{\sqrt{2\pi}}e^{ipx/\hbar}$$

BRACKETS: BRA'S AND KETS

- Inner product in matrix notation

$$\langle\alpha|\beta\rangle = \begin{pmatrix} a_1^* & a_2^* & \dots & a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + \dots a_n^* b_n$$

- "Bra" acts on the ket by row vector multiplication
- Now with possible infinite basis:

$$\langle\alpha| = \int \alpha^*(\dots)dx$$

PROJECTION AND IDENTITY OPERATORS

- Projection operator:

$$\hat{P} = |i\rangle\langle i|$$

- identity operator:

$$\hat{1} = \sum_n |i\rangle\langle i|$$