PHOT 301: Quantum Photonics

LECTURE 01

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INTRODUCTION

CLASSICAL VIEW

- Matter is described with **particles**
- Newton's equation:

$$\vec{F} = m\,\vec{a} = \frac{\partial^2 r_i}{\partial t^2}$$

- Forces act on **point** masses
- The force is the gradient of the **potential** energy:

$$\vec{F} = -\nabla V$$

- Conservation of energy $E = E_{kin} + E_{pot} = T + V$

CLASSICAL VIEW

- Light is described by **waves**
- Maxwell's equations

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ c^2 \nabla \times \vec{B} = \dfrac{\vec{J}}{\epsilon_0} + \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

CLASSICAL VIEW

- Light is described by **waves**
- Maxwell's equations
- If there are no charges or currents then $\rho = 0$ and $\vec{J} = 0$ then:

$$\begin{cases} \nabla^2 \vec{E} - \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \nabla^2 \vec{B} - \dfrac{1}{c^2}\dfrac{\partial^2 \vec{B}}{\partial t^2} = 0 \end{cases}$$

Vector components $u := E_i,\ B_i$ obey the wave equation:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial u}{\partial t^2} = 0$$

CLASSICAL VIEW

Light and matter are treated different

- Light has wave-like behavior
- Matter exists of particles

Problems:

- Hydrogen atom: Electron should fall on nuclues
- Specific energy bands of atomic spectra?
- Electrons can tunnel **through** potential energy barriers

Quantum mechanics combines both
And solves everything?

THE SCHRÖDINGER EQUATION

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

where

- Complex wave function: $\Psi \to \Psi(x, y, z, t)$
- Laplacian $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial^2 x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}$
- potential energy: $V \to V(x, y, z, t)$
- $\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}$ J s

THE SCHRÖDINGER EQUATION

We will first consider 1D problems:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t)$$

- The complex wave function $\Psi(x,t)$ is not observable
- Probability to find particle in x at time t given by $|\Psi(x,t)|^2$:

$$P(x\in[a,b]) = \int_a^b |\Psi(x,t)|^2 dx$$

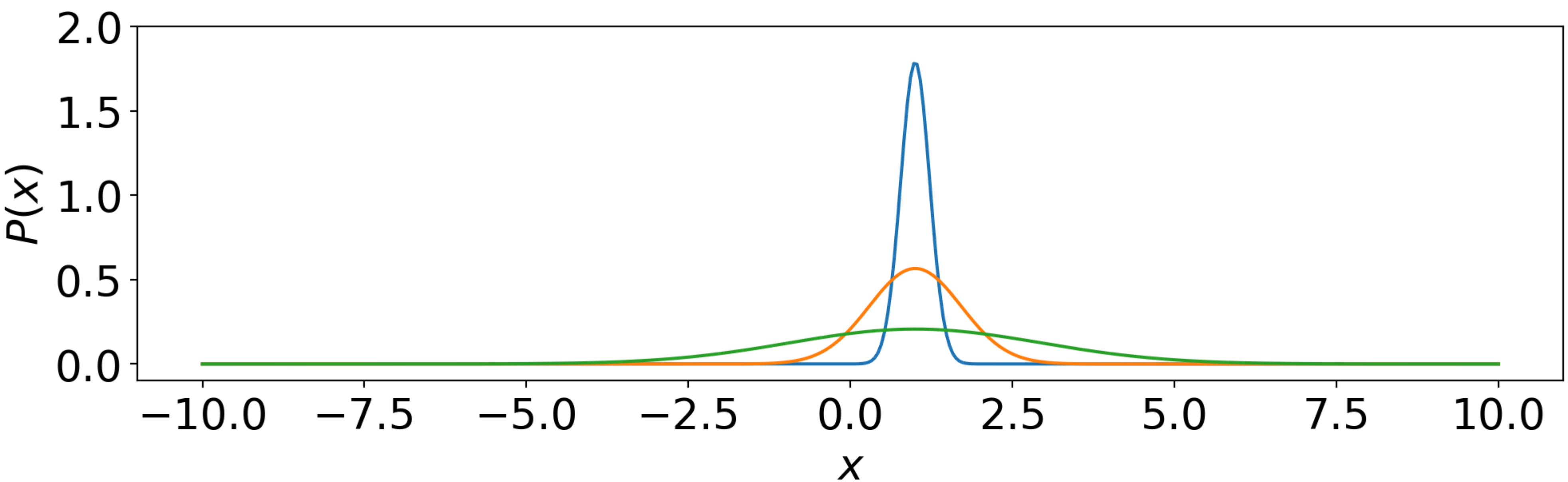
PROBABILISTIC VIEW

$$P(x \in [a,b]) = \int_a^b |\Psi(x,t)|^2 dx$$

PROBABILISTIC VIEW: MEASEREMENT PROBLEM

Copenhagen interpretation:

- Before measurement: probability according to $|\Psi(x,t)|^2$
- Measurement: Wave function collapses to a single state $\longrightarrow$ δ-function
- After measurement: δ-function spreads out again over time.

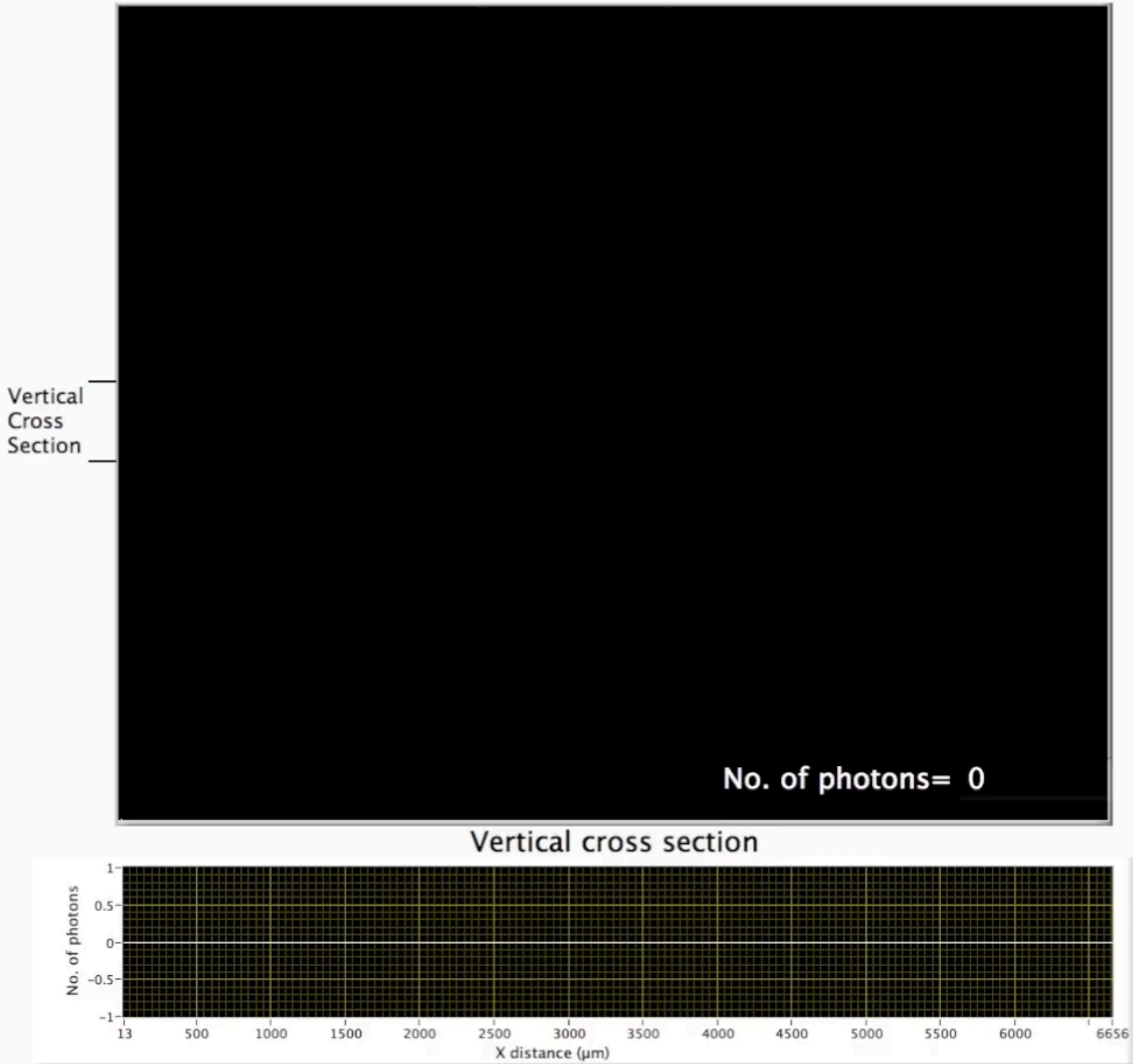


DOUBLE SLIT EXPERIMENTS:

Typical thought-experiment

- What happens if electron “particles” are fired through a double slit?
- What happens if light at low intensity (single photons) is used?

'Ghost' Young's double slit with a coherent source photon by photon



PROBABILITY AND EXPECTATION VALUES

Probability density function $\rho(x)$:

$$\text{Expectation value of } x \qquad \langle x \rangle = \int_{-\infty}^{\infty} x\, \rho(x)\, dx$$

$$\text{Expectation value of } f(x) \qquad \langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\, \rho(x)\, dx$$

PROBABILITY AND EXPECTATION VALUES

Probability density function $\rho(x)$:

Expectation value of x $\qquad \langle x \rangle = \int_{-\infty}^{\infty} x\,\rho(x)\,dx$

Expectation value of $f(x)$ $\qquad \langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\,\rho(x)\,dx$

Variance σ^2 $\qquad \langle (\Delta x)^2 \rangle = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2\,\rho(x)\,dx$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

Standard deviation $\qquad \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

NORMALIZATION OF THE WAVE FUNCTION

- $|\Psi(x,t)|^2$ is like the probability density $\rho(x)$
- The total probability to find a particle **somewhere** must be **one**:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

So wave function $\Psi(x,t)$:

- Is a solution of the Schrodinger equation
- Must be normalizable

$$\Longleftrightarrow \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx \text{ exists and is finite}$$

NORMALIZATION OF THE WAVE FUNCTION

If $\Psi(x,t)$ is normalized at $t=0$ then it is always normalized.

Follows from the Schrodinger equation (see Griffiths page 15):

$$\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx = 0$$

EXPECTATION VALUES

What are the particle's:

- position x?
- velocity v ? or
- momentum $p = mv$?

Calculate the expectation (average) values:

$$\text{Expectation value of } x \qquad \langle x \rangle = \int_{-\infty}^{\infty} x \, |\Psi(x,t)|^2 \, dx$$

$$\text{Expectation value of } p = mv \qquad m \frac{d\langle x \rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} \, dx$$

POSITION AND MOMENTUM OPERATORS

Expectation values are calculated as

$$\text{Position } x \qquad \langle x \rangle = \int_{-\infty}^{\infty} x \left|\Psi(x,t)\right|^2 dx = \int_{-\infty}^{\infty} \Psi^* \, [x] \, \Psi \, dx$$

$$\text{Momentum } p \quad m\frac{d\langle x \rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx = \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar \frac{\partial}{\partial x}\right] \Psi \, dx$$

$$\text{Position operator} \qquad \hat{x} = x$$

$$\text{Momentum operator} \qquad \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

SCHRODINGER EQUATION WITH OPERATORS

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x}\Psi + V\Psi$$

We have the operators:

$$\text{Position operator} \qquad \hat{x} = x$$

$$\text{Momentum operator} \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x}$$

Using operators in the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{1}{2m}[-i\hbar\frac{\partial}{\partial x}]^2\Psi + V\Psi = \frac{\hat{p}^2}{2m}\Psi + V\Psi = (\hat{T} + \hat{V})\Psi = \hat{\mathcal{H}}\Psi$$

CORRESPONDENCE PRINCIPLE

- Large systems: **Quantum mechanics** $\longrightarrow$ **classical physics**
- Ehrenfest's theorem:

$$m\frac{d}{dt}\langle x\rangle, \qquad \frac{d}{dt}\langle p\rangle = -\langle\frac{\partial V(x)}{\partial x}\rangle$$

UNCERTAINTY RELATION: POSITION VS. MOMENTUM

- de Broglie relation

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda} \quad (= \hbar k)$$

- Think about a Gaussian wave pulse in Fourier analysis
  - Sharp pulses in space are spread out in (momentum) k-space
  - Sharp pulses in k-space are spread out in space
- Uncertainty of position vs. momentum

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

SUMMARY

- Quantum mechanics is governed by the Schrodinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

- Similar to our standard wave equation
- But the wave function $\Psi(x, y, z, t)$ is complex-valued
- Probability density to find a particle $|\Psi(x,t)|^2 = \Psi^*(x,t)\,\Psi(x,t)$
- “Real” quantities and measurements represented by operators acting on the wave function