

PHOT 301: Quantum Photonics

LECTURE 01

Michaël Barbier, Summer (2024-2025)

INTRODUCTION

OVERVIEW

week	Topic	Reading
Week 1	Introduction & Required Mathematical Methods. Waves and Schrödinger's equation, Probability, Uncertainty and Time evolution. Infinite square well.	Ch. 1 & Ch. 2 (up to the infinite well)
Week 2	The harmonic oscillator, Creation and annihilation operators. Free particle, 1D Bound states & Scattering/Transmission, Finite well	
Week 3	Quantum mechanics formalism: Functions and operators, uncertainty. Approximation methods.	
Week 4	Angular momentum and the Hydrogen atom, Spin Magnetic fields, The Pauli equation, Minimal Coupling, Aharonov Bohm Perturbation: Fine Structure of Hydrogen, The Zeeman Effect	
Week 5	Identical particles, Periodic table, Molecular bonds, Periodic structures, Band structure, Bloch functions Time-dependent perturbation: Absorption, spontaneous emission, and stimulated emission	
Week 6	Final exam	

FOR NEXT WEEK

- Textbook Chapter 1: 1.1, 1.2, 1.3, 1.5, 1.8
- Textbook Chapter 2: 2.1(c), 2.3, 2.4, 2.5, 2.7
- Homework documents:
 - phot301_homework_integration.pdf
 - phot301_homework_solving_equations.pdf
 - phot301_homework_fourier.pdf
- Reading (by Thursday 24 July 2025): Chapter 2 of Griffiths

CLASSICAL VIEW

- Matter is described with **particles**
- Newton's equation:

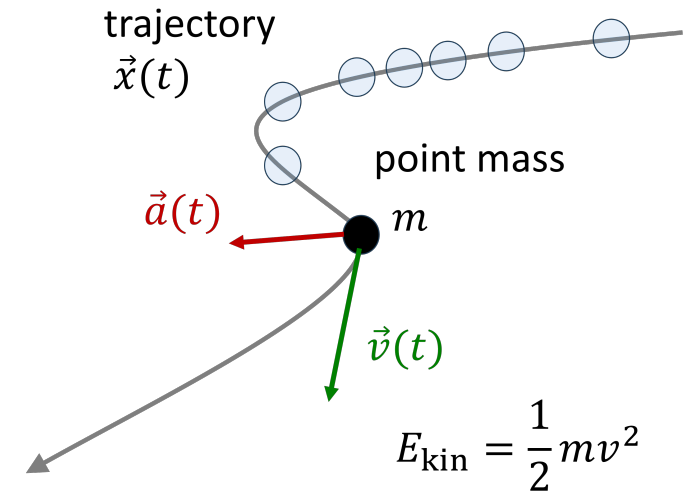
$$\vec{F} = m \vec{a} = m \frac{\partial^2 \vec{r}}{\partial t^2}$$

- Forces act on **point** masses
- The force is the gradient of the **potential** energy:

$$\vec{F} = -\nabla V$$

- Conservation of energy

$$E = E_{kin} + E_{pot} = T + V$$



CLASSICAL VIEW

- Light is described by **waves**
- Maxwell's equations

$$\left\{ \begin{array}{l} \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ c^2 \nabla \times \vec{B} = \frac{\vec{J}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t} \end{array} \right.$$

CLASSICAL VIEW

- Light is described by **waves**
- Maxwell's equations
- If there are no charges or currents then $\rho = 0$ and $\vec{J} = 0$ then:

$$\begin{cases} \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \end{cases}$$

Vector components $u := E_i, B_i$ obey the wave equation:

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

CLASSICAL VIEW

Light and matter are treated different

- Light has wave-like behavior
- Matter exists of particles

Problems:

- Hydrogen atom: Electron should fall on nuclues
- Specific energy bands of atomic spectra?
- Electrons can tunnel **through** potential energy barriers

Quantum mechanics combines both
And solves everything?

THE SCHRÖDINGER EQUATION

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi$$

where

- Complex wave function: $\Psi \rightarrow \Psi(x, y, z, t)$
- Laplacian $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- potential energy: $V \rightarrow V(x, y, z, t)$
- $\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} \text{ J s}$

THE SCHRÖDINGER EQUATION

We will first consider 1D problems:

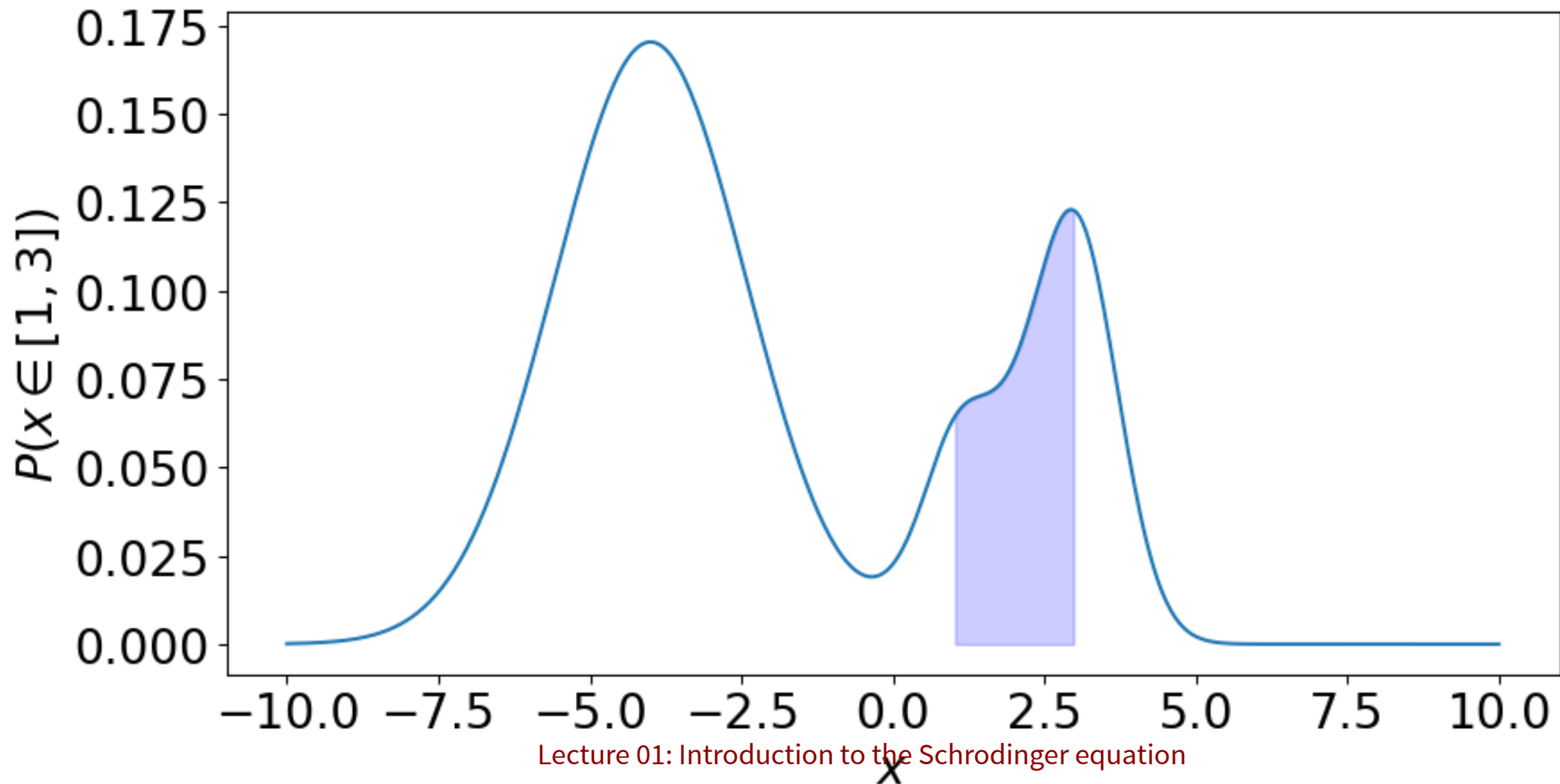
$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V(x, t) \Psi(x, t)$$

- The complex wave function $\Psi(x, t)$ is not observable
- Probability to find particle in x at time t given by $|\Psi(x, t)|^2$:

$$P(x \in [a, b]) = \int_a^b |\Psi(x, t)|^2 dx$$

PROBABILISTIC VIEW

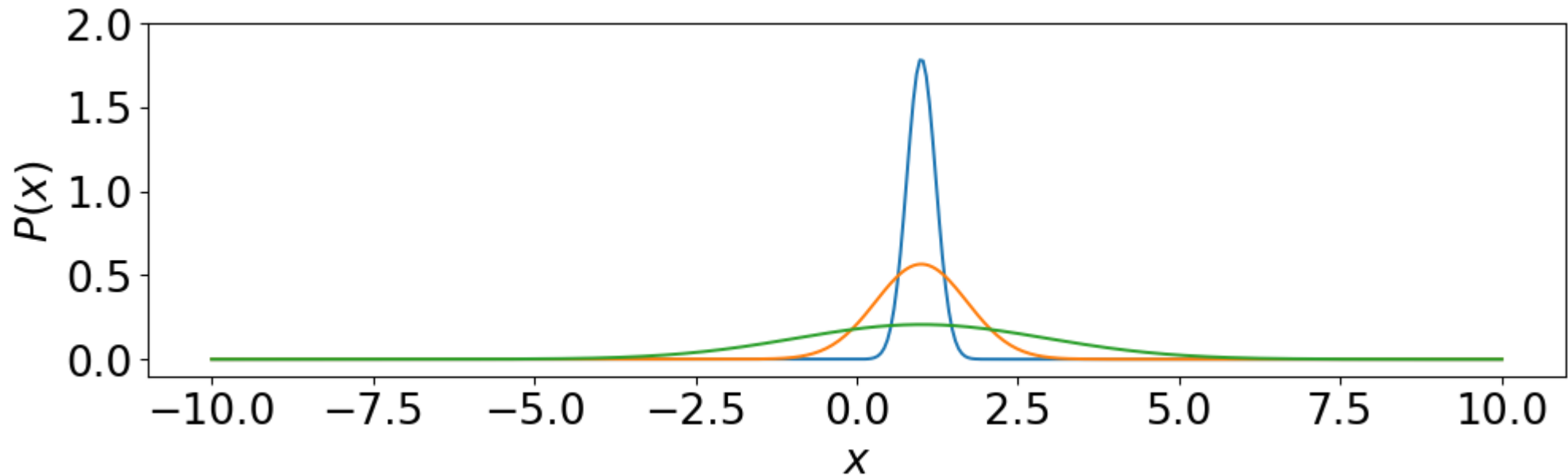
$$P(x \in [a, b]) = \int_a^b |\Psi(x, t)|^2 dx$$



PROBABILISTIC VIEW: MEASUREMENT PROBLEM

Copenhagen interpretation:

- Before measurement: probability according to $|\Psi(x, t)|^2$
- Measurement: Wave function collapses to a single state $\longrightarrow \delta$ -function
- After measurement: δ -function spreads out again over time.

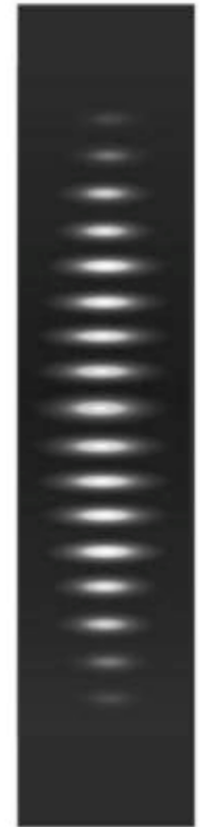
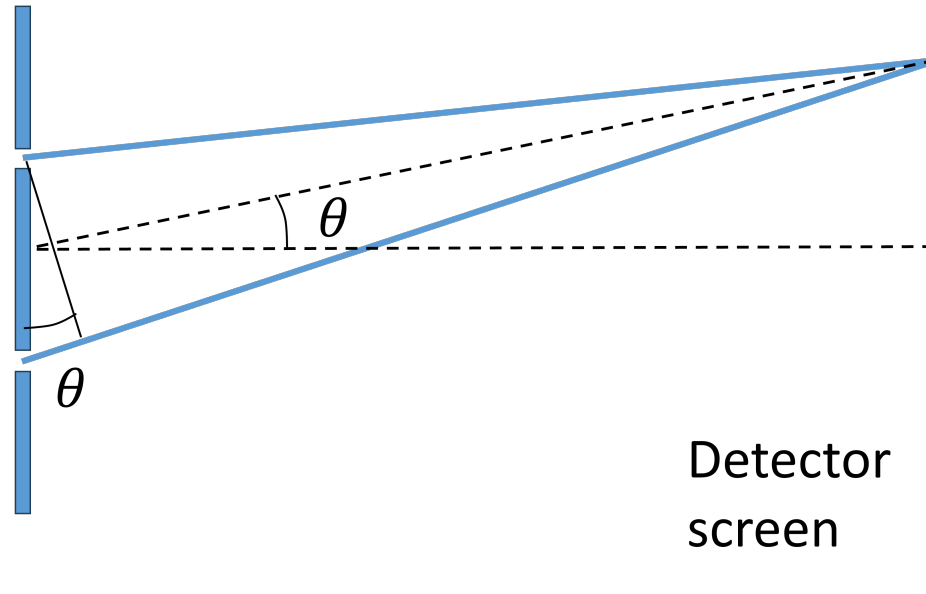
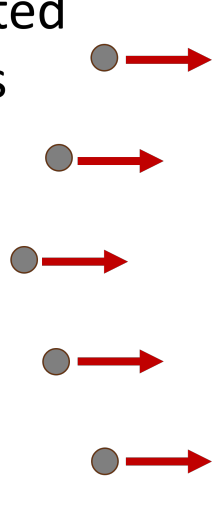


DOUBLE SLIT EXPERIMENTS:

Typical thought-experiment

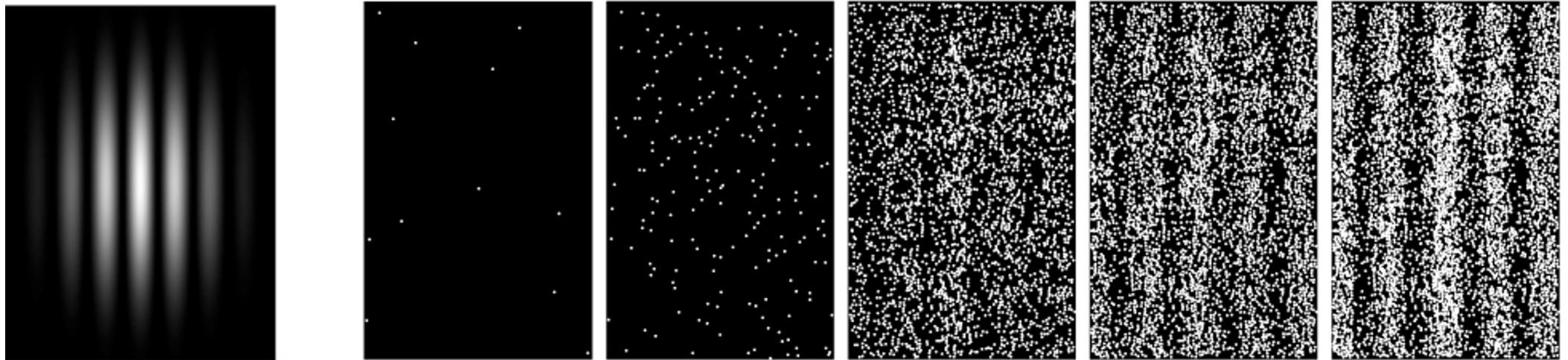
- What happens if electron “particles” are fired through a double slit?
- What happens if light at low intensity (single photons) is used?

Accelerated
electrons



DOUBLE SLIT EXPERIMENTS:

- Electrons arrive one by one
- probability to find an electron at coordinates $(x, y, z) \propto |\Psi(x, y, z)|^2$
- Higher electron densities diffraction pattern: $\propto |\Psi(x, y, z)|^2$
- *Similar* to the intensity of electric waves: $I \propto E^2$



PROBABILITY AND EXPECTATION VALUES

Probability density function $\rho(x)$:

Expectation value of x

$$\langle x \rangle = \int_{-\infty}^{\infty} x \rho(x) dx$$

Expectation value of $f(x)$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x) \rho(x) dx$$

PROBABILITY AND EXPECTATION VALUES

Probability density function $\rho(x)$:

Expectation value of x $\langle x \rangle = \int_{-\infty}^{\infty} x \rho(x) dx$

Expectation value of $f(x)$ $\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x) \rho(x) dx$

Variance σ^2 $\langle (\Delta x)^2 \rangle = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 \rho(x) dx$
 $= \langle x^2 \rangle - \langle x \rangle^2$

Standard deviation $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

NORMALIZATION OF THE WAVE FUNCTION

- $|\Psi(x, t)|^2$ is like the probability density $\rho(x)$
- The total probability to find a particle **somewhere** must be **one**:

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1$$

So wave function $\Psi(x, t)$:

- Is a solution of the Schrodinger equation
- Must be normalizable

$$\iff \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx \text{ exists and is finite}$$

NORMALIZATION OF THE WAVE FUNCTION

If $\Psi(x, t)$ is normalized at $t = 0$ then it is always normalized.

Follows from the Schrodinger equation (see Griffiths page 15):

$$\frac{d}{dt} \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 0$$

EXPECTATION VALUES

What are the particle's:

- position x ?
- velocity v ? or
- momentum $p = mv$?

Calculate the expectation (average) values:

Expectation value of x $\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x, t)|^2 dx$

Expectation value of $p = mv$ $m \frac{d\langle x \rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx$

POSITION AND MOMENTUM OPERATORS

Expectation values are calculated as

$$\text{Position } x \quad \langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x, t)|^2 dx = \int_{-\infty}^{\infty} \Psi^* [x] \Psi dx$$

$$\text{Momentum } p \quad m \frac{d\langle x \rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx = \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \Psi dx$$

$$\text{Position operator} \quad \hat{x} = x$$

$$\text{Momentum operator} \quad \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

SCHRODINGER EQUATION WITH OPERATORS

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi + V\Psi$$

We have the operators:

$$\text{Position operator} \quad \hat{x} = x$$

$$\text{Momentum operator} \quad \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

Using operators in the Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi = \frac{1}{2m} \left[-i\hbar \frac{\partial}{\partial x} \right]^2 \Psi + V\Psi = \frac{\hat{p}^2}{2m} \Psi + V\Psi = (\hat{T} + \hat{V})\Psi = \hat{\mathcal{H}}\Psi$$

CORRESPONDENCE PRINCIPLE

- Large systems: **Quantum mechanics** \longrightarrow **classical physics**
- Ehrenfest's theorem:

$$m \frac{d}{dt} \langle x \rangle, \quad \frac{d}{dt} \langle p \rangle = - \left\langle \frac{\partial V(x)}{\partial x} \right\rangle$$

UNCERTAINTY RELATION: POSITION VS. MOMENTUM

- de Broglie relation

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda} \quad (= \hbar k)$$

- Think about a Gaussian wave pulse in Fourier analysis
 - Sharp pulses in space are spread out in (momentum) k-space
 - Sharp pulses in k-space are spread out in space
- Uncertainty of position vs. momentum

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

SUMMARY

- Quantum mechanics is governed by the Schrodinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi$$

- Similar to our standard wave equation
- But the wave function $\Psi(x, y, z, t)$ is complex-valued
- Probability density to find a particle $|\Psi(x, t)|^2 = \Psi^*(x, t) \Psi(x, t)$
- “Real” quantities and measurements represented by operators acting on the wave function