PHOT 301: Quantum Photonics

Quiz 3: questions & solutions

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Exam questions

Grading: Each quiz counts for 7.5% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam. Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is weighed equally.
- Answers should contain: The final formula/expression together with its derivation.

Question 1: Operators and Commutators

(a) Calculate the commutator $[1/x, \hat{p}]$.
(b) Show that e^{-x^2} is an eigenstate of the operator $\hat{Q} = -\frac{1}{x}\frac{d}{dx}$. Calculate its eigenvalue.

Solution (Q1)

(a) We fill in the definition of the momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$ and apply the operator to a test function $f \equiv f(x)$:

$$\begin{aligned}
[1/x, \hat{p}]f &= \left(\frac{1}{x}\hat{p} - \hat{p}\frac{1}{x}\right) f \\
&= -i\hbar\left(\frac{1}{x}\frac{df}{dx} - \frac{d\frac{f}{x}}{dx}\right) \\
&= -i\hbar\left(\frac{1}{x}\frac{df}{dx} + \frac{f}{x^2} - \frac{1}{x}\frac{df}{dx}\right) \\
&= -i\hbar\frac{f}{x^2}
\end{aligned}$$

Removing the test function gives $[1/x, \hat{p}] = \frac{-i\hbar}{x^2}$.

(b) If e^{-x^2} is an eigenfunction then $\hat{Q}e^{-x^2} = qe^{-x^2}$ with q a constant, the eigenvalue:

$$\hat{Q}e^{-x^2} = -\frac{1}{x}\frac{d\,e^{-x^2}}{dx} = -\frac{1}{x}(-2x)e^{-x^2} = 2e^{-x^2}$$

Therefore, the eigenvalue $q = 2$ and e^{-x^2} is indeed an eigenfunction.

Question 2: Operators in finite dimensional space

Assume that the Hamiltonian operator $\hat{H}$ of a two-level system is represented by the following matrix:

$$H = \begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Solve the (time-independent) eigenvalue equation $H|\psi\rangle = E|\psi\rangle$ to obtain eigenenergies E_n of the system.
(b) Then calculate the normalized eigenstates $|\psi_n\rangle$. *Hint:* The eigenstates should be superpositions of the basis vectors $|1\rangle$ and $|2\rangle$.

Solution (Q2)

(a) To find the eigenenergies we solve the eigenvalue equation:

$$\begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

Where the eigenvectors are represented by so far unknown vectors: $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$. We follow the standard procedure to solve such an eigenvalue equation, putting the determinant $\det(\lambda\mathbb{1} - H) = 0$:

$$0 = \det\left[\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} - \begin{pmatrix}1 & i\sqrt{3}\\ -i\sqrt{3} & -1\end{pmatrix}\right] = \begin{vmatrix}\lambda-1 & -i\sqrt{3}\\ i\sqrt{3} & \lambda+1\end{vmatrix} = (\lambda-1)(\lambda+1)-3 = \lambda^2-4 = (\lambda-2)(\lambda+2)$$

Therefore the eigenvalues are $E_\pm \equiv \lambda_\pm = \pm 2$.

(b) To obtain the eigenvectors/eigenstates we fill in the eigenvalues in the original equation:

$\boxed{E_+ = 2}$:

$$\begin{pmatrix}1 & i\sqrt{3}\\ -i\sqrt{3} & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 2\begin{pmatrix}x\\ y\end{pmatrix} \quad \Rightarrow \quad x + i\sqrt{3}y = 2x \quad \Rightarrow \quad x = i\sqrt{3}y$$

This results in eigenvector $|\psi_+\rangle = \begin{pmatrix}x\\ y\end{pmatrix} = A_+\begin{pmatrix}i\sqrt{3}\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}i\sqrt{3}\\ 1\end{pmatrix}$.

Here we calculated the normalization constant $A_+ = 1/\sqrt{\langle\psi_+|\psi_+\rangle} = 1/\sqrt{3+1} = 1/2$.

$\boxed{E_- = -2}$:

$$\begin{pmatrix}1 & i\sqrt{3}\\ -i\sqrt{3} & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -2\begin{pmatrix}x\\ y\end{pmatrix} \quad \Rightarrow \quad x + i\sqrt{3}y = -2x \quad \Rightarrow \quad y = i\sqrt{3}x$$

This results in eigenvector $\psi_- = \begin{pmatrix}x\\ y\end{pmatrix} = A_-\begin{pmatrix}1\\ i\sqrt{3}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\ i\sqrt{3}\end{pmatrix}$. Here we calculated the normalization constant $A_- = 1/\sqrt{\langle\psi_-|\psi_-\rangle} = 1/\sqrt{1+3} = 1/2$.

Question 3: Dirac notation

Consider the 3-dimensional space with orthonormal basis: $\{|1\rangle, |2\rangle, |3\rangle\}$. Further, kets $|\alpha\rangle$ and $|\beta\rangle$ are given by:

$$|\alpha\rangle = i|1\rangle + |2\rangle - |3\rangle, \qquad |\beta\rangle = |1\rangle + |2\rangle$$

(a) Show that $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$.
(b) Compute the operator $\hat{A} = |\alpha\rangle\langle\beta|$. Represent the operator as a 3×3 matrix in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis and calculate its elements.

Solution (Q3)

(a) We calculate first the "bra" version of each ket:

$$\langle\alpha| = -i\langle 1| + \langle 2| - \langle 3|, \qquad \langle\beta| = \langle 1| + \langle 2|$$

Then we calculate both inner products:

$$\begin{aligned}
\langle\alpha|\beta\rangle &= (-i\langle 1| + \langle 2| - \langle 3|)\,(|1\rangle + |2\rangle) \\
&= -i\langle 1|1\rangle - i\langle 1|2\rangle + \langle 2|1\rangle + \langle 2|2\rangle - \langle 3|1\rangle + \langle 3|2\rangle \\
&= -i - 0 + 0 + 1 - 0 + 0 = 1 - i \\
\langle\beta|\alpha\rangle &= (\langle 1| + \langle 2|)\,(i|1\rangle + |2\rangle - |3\rangle) \\
&= i\langle 1|1\rangle + \langle 1|2\rangle - \langle 1|3\rangle + i\langle 2|1\rangle + \langle 2|2\rangle - \langle 2|3\rangle \\
&= i + 0 - 0 + 0 + 1 - 0 = 1 + i
\end{aligned}$$

Therefore $\langle\alpha|\beta\rangle = 1 - i = (1+i)^* = \langle\beta|\alpha\rangle^*$

Alternatively we can use the representation in $\mathbb{C}^2$ where :

$$|\alpha\rangle = \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix}, \quad |\beta\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \langle\alpha| = |\alpha\rangle^\dagger = \begin{pmatrix} -i & 1 & -1 \end{pmatrix}, \quad \langle\beta| = |\beta\rangle^\dagger = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$

and the inner products can be written as simple matrix products:

$$\langle\beta|\alpha\rangle = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix} = i + 1 + 0 = 1 + i$$

$$\langle\alpha|\beta\rangle = \begin{pmatrix} -i & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -i + 1 + 0 = 1 - i$$

Giving the same results.

(b) To calculate the matrix elements we can use the vector representation of $|\alpha\rangle$ and $\langle\beta|$ as above and calculate the outer product:

$$\hat{A} = |\alpha\rangle\langle\beta| = \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} i & i & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$