PHOT 301: Quantum Photonics

Quiz 2: questions & solutions

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Exam questions

Grading: Each quiz counts for 7.5% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam. Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is weighed equally.
- Answers should contain: The final formula/expression together with its derivation.

Question 1: Ladder operators

Consider the following wave function $\psi(x)$ in a quantum harmonic oscillator.

$$\psi(x) = A\left(2\,\hat{a}_+\,\psi_0 + \hat{a}_-\,(\psi_1 + \psi_0)\right), \quad \text{where} \qquad \hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad \hat{a}_-\psi_n = \sqrt{n}\psi_{n-1}$$

with A a normalization constant, $\hat{a}_\pm$ the ladder operators, and ψ_n the eigenstates.

(a) Simplify the expression for $\psi(x)$ by applying the ladder operators. The resulting expression shouldn't contain any ladder operators.
(b) Then calculate the normalization constant A of the wave function.

Solution (Q1)

(a) The wave function can be simplified by using $\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1}$ and $\hat{a}_-\psi_n = \sqrt{n}\psi_{n-1}$:

$$\psi(x) = A\left(2\hat{a}_+\psi_0 + \hat{a}_-\psi_1 + \hat{a}_-\psi_0\right) = A\left(2\sqrt{1}\,\psi_1 + \sqrt{1}\,\psi_0 + 0\right) = A\left(2\psi_1 + \psi_0\right)$$

(b) The total probability should be equal to one:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(x)|^2\,dx = |A|^2 \int_{-\infty}^{\infty} |2\psi_1 + \psi_0|^2\,dx \\
&= |A|^2 \int_{-\infty}^{\infty} \left[4|\psi_1|^2 + |\psi_0|^2 + 4\Re\{\psi_1^*\psi_0\}\right]\,dx \\
&= |A|^2 \int_{-\infty}^{\infty} 4|\psi_1|^2\,dx + |A|^2 \int_{-\infty}^{\infty} |\psi_0|^2\,dx + 4|A|^2\,\Re\left\{\int_{-\infty}^{\infty} \psi_1^*\psi_0\,dx\right\} \\
&= |A|^2(4+1+0) = 5|A|^2
\end{aligned}$$

Where the last term is zero since the eigenstates ψ_n are orthonormal. Thus $A = \frac{1}{\sqrt{5}}$ when we choose it real and positive.

Question 2: Time evolution

A particle in a harmonic oscillator has following normalized wave function at time zero:

$$\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_0 + \psi_1), \quad \text{with} \quad \begin{cases} \psi_0(x) = \pi^{-1/4}\sqrt{\beta}e^{-\beta^2x^2/2} \\ \psi_1(x) = \sqrt{2}\pi^{-1/4}\beta^{3/2}xe^{-\beta^2x^2/2} \end{cases}, \quad \text{and} \quad \beta = \sqrt{\frac{m\omega}{\hbar}}$$

(a) Write down the expression for the time-dependent wave function $\Psi(x,t)$.
(b) Calculate expectation value $\langle x \rangle$. Show that it oscillates around $x = 0$ in time. *Hint:* Only fill in the explicit functions for the eigenstates ψ_n in the end.

Solution (Q2)

(a) The time-dependent function can be expressed as:

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}(\psi_0 e^{-iE_0t/\hbar} + \psi_1 e^{-iE_1t/\hbar}) \\
&= \frac{1}{\sqrt{2}}(\psi_0 e^{-i\omega t/2} + \psi_1 e^{-i3\omega t/2})
\end{aligned}$$

Where we filled in the energy $E_0 = \frac{1}{2}\hbar\omega$ of the ground state and $E_1 = \frac{3}{2}\hbar\omega$ of the first excited state.

(b) The (time-dependent) expectation value of the position is $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx$:

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x\,\left|\psi_0 e^{-i\omega t/2} + \psi_1 e^{-i3\omega t/2}\right|^2\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x\,\left(|\psi_0|^2 + |\psi_1|^2 + \psi_0\psi_1 e^{-i\omega t} + \psi_0\psi_1 e^{i\omega t}\right)\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x\,\left(|\psi_0|^2 + |\psi_1|^2 + 2\psi_0\psi_1\cos(\omega t)\right)\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x\,|\psi_0|^2\,dx + \frac{1}{2}\int_{-\infty}^{\infty} x\,|\psi_1|^2\,dx + \int_{-\infty}^{\infty} x\,\psi_0\psi_1\cos(\omega t)\,dx
\end{aligned}$$

Here we used the fact that the eigenstates are real-valued ($\psi_n^* = \psi_n$). Because of symmetry the first two integrals are zero. Afterwards we fill in the wave functions ψ_0 and ψ_1.

$$\begin{aligned}
\langle x\rangle &= 0 + 0 + \cos(\omega t)\int_{-\infty}^{\infty} x\,\psi_0\psi_1\,dx \\
&= \cos(\omega t)\int_{-\infty}^{\infty} x\,\left(\pi^{-1/4}\sqrt{\beta}e^{-\beta^2x^2/2}\right)\left(\sqrt{2}\pi^{-1/4}\beta^{3/2}xe^{-\beta^2x^2/2}\right)\,dx \\
&= \cos(\omega t)\sqrt{2}\beta^2\pi^{-1/2}\int_{-\infty}^{\infty} x^2 e^{-\beta^2x^2}\,dx \\
&= \cos(\omega t)\sqrt{2}\beta^2\pi^{-1/2}\frac{\sqrt{\pi}}{2\beta^3} \\
&= \frac{1}{\sqrt{2}\beta}\cos(\omega t)
\end{aligned}$$

Question 3: Delta-function potential well

A particle with mass m in an delta-function potential well with strength α, that is $V(x) = -\alpha\,\delta(x)$ has the following normalized wave function (defined with $x \in \mathbb{R}$):

$$\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}, \qquad \text{with} \quad \kappa = \frac{m\alpha}{\hbar^2}, \quad \text{and} \quad E = -\frac{m\alpha^2}{2\hbar^2}$$

(a) Calculate the probability P for the particle to be found inside interval $[-1/\kappa,\, 1/\kappa]$.
(b) Suppose the well is now three times as strong, that is $\alpha_{\text{new}} = 3\alpha$. Calculate the "new" energy and compare with the original one: $E_{\text{new}} \leftrightarrow E$?

Solution (Q3)

(a) The probability $P(-1/\kappa \leq x \leq 1/\kappa)$:

$$
\begin{aligned}
P &= \int_{-1/\kappa}^{1/\kappa} |\psi|^2 \, dx \\
&= \int_{-1/\kappa}^{1/\kappa} \kappa e^{-2\kappa|x|} \, dx \\
&= 2 \int_{0}^{1/\kappa} \kappa e^{-2\kappa x} \, dx \\
&= 2\kappa \, \frac{1}{-2\kappa} \left(e^{-2\kappa x}\right) \Big|_0^{1/\kappa} \\
&= 1 - e^{-2}
\end{aligned}
$$

which is in interval $[0, 1]$ as should, and since $e = 2.718 \gtrapprox 2.7$, the probability to be inside the interval is $P \gtrapprox 1 - \frac{1}{2.7^2}$ is relatively large.

(b) The energy is given by $E = -\frac{m\alpha^2}{2\hbar^2}$. If the strength α increases threefold then the new energy is:

$$
E_{\text{new}} = -\frac{m(3\alpha)^2}{2\hbar^2} = -9 \, \frac{m\alpha^2}{2\hbar^2} = 9E
$$

which is 9 times larger but since the energy level is negative, "lower" than the original energy E.