PHOT 222: Quantum Photonics

Quiz 1: questions & solutions

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Exam questions

Grading: Each quiz counts for 7.5% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam. Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is weighed equally.
- Answers should contain: The final formula/expression together with its derivation.

Following hints were given at the end of the exam form (here we mean with "energy" E the expectation value of the energy or more correctly $\langle \hat{H} \rangle$):

Hints for questions 2 and 3: The expansion $\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$ has coefficients:

$$c_n = \int_0^L \psi_n^* \, \psi \, dx, \qquad \text{where} \qquad \sum_{n=1}^{\infty} |c_n|^2 = 1 \quad \text{and} \quad E = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

Question 1: Wave functions

Consider the wave function $\psi(x)$ defined with $x \in \mathbb{R}$:

$$\psi(x) = Axe^{-x^2/2}$$

with A a normalization constant.

(a) First calculate the normalization constant A of the wave function.
(b) Then calculate the expectation value $\langle x^2 \rangle$.

Solution (Q1)

(a) The total probability should be equal to one:

$$1 = |A|^2 \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = |A|^2 \int_{-\infty}^{\infty} |xe^{-x^2/2}|^2 \, dx = |A|^2 \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx$$
$$= |A|^2 \frac{\sqrt{\pi}}{2}$$

And thus $A = \sqrt{\frac{2}{\sqrt{\pi}}}$ when we choose it real and positive.

(b) The expectation value is defined as $\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi(x)^* \, x^2 \, \psi(x) \, dx$ leading to:

$$\langle x^2 \rangle = |A|^2 \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \, dx = |A|^2 \int_{-\infty}^{\infty} x^2 |xe^{-x^2/2}|^2 \, dx = |A|^2 \int_{-\infty}^{\infty} x^4 e^{-x^2} \, dx$$
$$= |A|^2 \frac{3\sqrt{\pi}}{4} = \frac{2}{\sqrt{\pi}} \frac{3\sqrt{\pi}}{4} = \frac{3}{2}$$

Question 2: Coefficient expansion

Assume a particle in an infinite well of width L has following wave function at time zero:

$$\Psi(x, 0) = \psi(x) = A \sin(\frac{\pi x}{L}) \cos(\frac{\pi x}{L})$$

(a) Calculate the normalization constant A.
(b) The wave function at time zero $\psi(x)$ can be expanded in the stationary states of the infinite well given by $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$. Prove that only one coefficient in the expansion $\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$ is nonzero (and calculate its value).

Solution (Q2)

First notice that the wave function can be rewritten (using goniometric identity $\sin(2\alpha) = 2 \sin \alpha \cos \alpha$) as:

$$\psi = A \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) = A \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right)$$

which corresponds to eigenstate $\psi_2(x)$ if we put $A \frac{1}{2} = \sqrt{2/L}$.

(a) The total probability should equal one:

$$1 = |A|^2 \int_0^L |\psi(x)|^2 \, dx = |A|^2 \int_0^L \frac{1}{4} \sin^2\left(\frac{2\pi x}{L}\right) dx = |A|^2 \frac{L}{4} \int_0^1 \sin^2(2\pi u) \, du = |A|^2 \frac{L}{8},$$

where we used substitution $u = x/L$. Therefore $A = \sqrt{8/L}$, if we choose A to be real and positive.

(b) The coefficients c_n in the expansion of the wavefunction $\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$ are given by:

$$\begin{aligned}
c_n &= \int_0^L \psi_n^*(x)\, \psi(x)\, dx \\
&= \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{8}{L}} \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) dx \\
&= \int_0^L \psi_n^*\, \psi_2\, dx \\
&= 1 \quad \text{if } n = 2, \quad \text{otherwise } 0
\end{aligned}$$

Where the last equality comes from the fact that the eigenstates $\psi_n(x)$ are orthonormal. Therefore the only nonzero coefficient is $c_2 = 1$:

$$c_1 = 0, \quad c_2 = 1, \quad c_3 = 0, \quad c_4 = 0, \quad c_5 = 0, \quad \ldots$$

Question 3: Infinite well

Assume a particle in an infinite well is in a superposition state $\psi(x) = c_2\psi_2(x) + c_3\psi_3(x)$ (at time equal to zero) and has energy $E = 6E_1^{\infty}$, where $E_1^{\infty} = \frac{\hbar^2\pi^2}{2mL^2}$.

(a) Express the energy values E_2 and E_3 in energy units of E_1^{∞}.
(b) Calculate the values of the coefficients c_2 and c_3.

Solution (Q3)

(a) The energy of the nth state in an infinite well is given by the formula:

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL} = n^2 E_1^{\infty}$$

Therefore $E_2 = 4E_1^{\infty}$ and $E_3 = 9E_1^{\infty}$.

(b) The expectation value of the energy $\langle \hat{H} \rangle = 6E_1^{\infty}$ is given by $\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$ where the coefficients need to fulfill the condition $1 = \sum_{n=1}^{\infty} |c_n|^2$. Therefore for $\psi(x) = c_2\psi_2(x) + c_3\psi_3(x)$ we end up with the system of equations to solve:

$$\begin{cases} E = 6E_1^{\infty} = |c_2|^2 E_2 + |c_3|^2 E_3 \\ 1 = |c_2|^2 + |c_3|^2. \end{cases}$$

Then fill in the values $E_2 = 4E_1^{\infty}$, and $E_3 = 9E_1^{\infty}$

$$\begin{cases} 6E_1^\infty = \left(4|c_2|^2 + 9|c_3|^2\right) E_1^\infty \\ |c_2|^2 = 1 - |c_3|^2. \end{cases}$$

and solve the system:

$$\begin{cases} 6 = \left(4\left(1 - |c_3|^2\right) + 9|c_3|^2\right) \\ |c_2|^2 = 1 - |c_3|^2. \end{cases} \Rightarrow \begin{cases} 2 = 5|c_3|^2 \\ |c_2|^2 = 1 - |c_3|^2. \end{cases} \Rightarrow \begin{cases} |c_3| = \sqrt{2/5} \\ |c_2| = \sqrt{3/5}. \end{cases}$$

If we choose the coefficients real and positive then $c_2 = \sqrt{3/5}$ and $c_3 = \sqrt{2/5}$.