PHOT 301: Quantum Photonics

Homework problems 4

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Problems

Here we list the problems with there final solutions so you can check whether you have the corrects answers. Some problems ask you to prove a theorem, for these problems, I write just some extra hints. The problems are from Griffiths 3rd edition. The problems for this week:

- Textbook Chapter 4: 4.1, 4.2 (you can check with the slides), 4.5, 4.13, 4.15, 4.17, 4.19, 4.21, 4.25, 4.26, 4.30, 4.34, 4.35, 4.37, 4.45, 4.52, 4.53, 4.54, 4.58

Problem 4.1

(a) Work out all of the **canonical commutation relations** for components of the operators $\hat{r}$ and $\hat{p}$: $[x, y]$, $[x, \hat{p}_y]$, $[x, \hat{p}_x]$, $[\hat{p}_y, \hat{p}_z]$, and so on. *Answer:*

$$[r_i, \hat{p}_j] = -[\hat{p}_i, r_j] = i\hbar\delta_{ij}, \quad [r_i, r_j] = [\hat{p}_i, \hat{p}_j] = 0,$$

where the indices stand for x, y, or z, and $r_x = x$, $r_y = y$, and $r_z = z$.

(b) Confirm the three-dimensional version of **Ehrenfest's theorem**,

$$\frac{d\langle\vec{r}\rangle}{dt} = \langle\hat{\vec{p}}\rangle, and \quad m\frac{d\langle\hat{\vec{p}}\rangle}{dt} = \langle -\nabla V\rangle.$$

(Each of these, of course, stands for three equations—one for each component.)
Hint: First check that the "generalized" Ehrenfest theorem, Equation 3.73, is valid in three dimensions.
(c) Formulate **Heisenberg's uncertainty principle** in three dimensions. *Answer:*

$$\sigma_x\sigma_{p_x} \geq \hbar/2, \qquad \sigma_y\sigma_{p_y} \geq \hbar/2, \qquad \sigma_z\sigma_{p_z} \geq \hbar/2,$$

but there is no restriction on, say, $\sigma_x\sigma_{p_y}$.

Solution (4.1)

(a) Use a test function $f \equiv f(x, y, z)$ to verify the commutator relations.
(b) Look at equation 3.62.

Problem 4.2

Use separation of variables in cartesian coordinates to solve the infinite cubical well (or "particle in a box"):

$$V(x, y, z) = \begin{cases} 0, x, y, z \text{ all between 0 and } a; \\ \infty, \text{otherwise} \end{cases}$$

(a) Find the stationary states, and the corresponding energies.
(b) Call the distinct energies E_1, E_2, E_3, ..., in order of increasing energy. Find E_1, E_2, E_3, E_4, E_5, and E_6. Determine their degeneracies (that is, the number of different states that share the same energy). *Comment:* In one dimension degenerate bound states do not occur (see Problem 2.44), but in three dimensions they are very common.

(c) What is the degeneracy of E_{14}, and why is this case interesting?

Solution (4.2)

(a) $\psi_{n_x,n_y,n_z} = \sqrt{8/a^3} \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{a} \sin \frac{n_z \pi z}{a}$ and $E_{n_x,n_y,n_z} = \frac{\hbar^2 \pi^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2) = (n_x^2 + n_y^2 + n_z^2)E_1^{\infty}$.

(b) Expressing energies E_n in units of $E_1^{\infty} = \frac{\hbar^2 \pi^2}{2ma^2}$:

$$E_1 = 3, \quad E_2 = 6, \quad E_3 = 9, \quad E_4 = 11, \quad E_5 = 12, \quad E_6 = 14.$$

Degeneracies (number of states with the same energy):

$$d_1 = 1, \quad d_2 = 3, \quad d_3 = 3, \quad d_4 = 3, \quad d_5 = 1, \quad E_6 = 6.$$

(c) $d_14 = 4$ whereas normally we expect 1 ($n_x = n_y = n_z$), 3 (only two quantum numbers the same), 6 (all three quantum numbers different $n_x \neq n_y \neq n_z$) from combinatorial reasoning. Check for yourself why this is.

Problem 4.5

Problem 4.5 Show that

$$\Theta(\theta) = A \ln[\tan(\theta/2)]$$

satisfies the θ equation (Equation 4.25), for $l = m = 0$. This is the unacceptable "second solution"—what's wrong with it?

Solution (4.5)

Θ blows up at $\theta = 0$ and $\theta = \pi$: $\Theta(\theta) \to \infty$ and $\Theta(\pi) \to \infty$. (show this)

Problem 4.15

(a) Find $\langle r \rangle$ and $\langle r^2 \rangle$ for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.

(b) Find $\langle x \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen. Hint: This requires no new integration—note that $r^2 = x^2 + y^2 + z^2$, and exploit the symmetry of the ground state.

(c) Find $\langle x^2 \rangle$ in the state $n = 2, l = 1, m = 1$. *Hint:* this state is not symmetrical in x, y, z. Use $x = r \sin\theta \cos\phi$.

Solution (4.15)

(a) $\langle r \rangle = \frac{3}{2}a$ and $\langle r^2 \rangle = 3a^2$.

(b) $\langle x \rangle = 0$ and $\langle x^2 \rangle = a^2$.

(c) $\langle x^2 \rangle = 12a^2$.

Problem 4.17

Calculate $\langle z\hat{H}z \rangle$, in the ground state of hydrogen. *Hint:* This takes two pages and six integrals, or four lines and no integrals, depending on how you set it up. To do it the quick way, start by noting that $[z, [\hat{H}, z]] = 2z\hat{H}z - \hat{H}z^2 - z^2\hat{H}$. The idea is to reorder the operators in such a way that $\hat{H}$ appears either to the left or to the right, because we know (of course) what $\hat{H}\psi_{100}$ is.

Solution (4.17)

$\langle z\hat{H}z \rangle = 0$.

Problem 4.19

A hydrogenic atom consists of a single electron orbiting a nucleus with Z protons. ($Z = 1$ would be hydrogen itself, $Z = 2$ is ionized helium, $Z = 3$ is doubly ionized lithium, and so on.) Determine the Bohr energies $E_n(Z)$, the binding energy $E_1(Z)$, the Bohr radius $a(Z)$, and the Rydberg constant $R(Z)$ for a hydrogenic atom. (Express your answers as appropriate multiples of the hydrogen values.) Where in the electromagnetic spectrum would the Lyman series fall (figure 4.10), for $Z = 2$ and $Z = 3$? *Hint:* There's nothing much to calculate here—in the potential (Equation 4.52) $e^2 \longrightarrow Ze^2$, so all you have to do is make the same substitution in all the final results.

Solution (4.19)

Bohr energies $E_n(Z) = Z^2 E_n$,
Binding energy $E1(Z) = Z^2 E_1$,
Bohr radius $a(Z) = a/Z$,
Rydberg constant $R(Z) = Z^2 R$. The Lyman series for $Z = 2$ falls in: $\lambda \in [22.8, 30.4]$ nm (ultraviolet).
The Lyman series for $Z = 3$ falls in: $\lambda \in [10.1, 13.5]$ nm (also ultraviolet).

Problem 4.21

The raising and lowering operators change the value of m by one unit:

$$L_+ f_l^m = (A_l^m)\, f_l^{m+1}, \qquad L_- f_l^m = (B_l^m)\, f_l^{m-1}$$

where A_l^m and B_l^m are constants. Question: What are they, if the eigenfunctions are to be normalized? Hint: First show that $L_\mp$ is the hermitian conjugate of $L_\pm$ (since L_x and L_y are observables, you may assume they are hermitian ... but prove it if you like); then use Equation 4.112. Answer:

$$A_l^m = \hbar\sqrt{l(l+1) - m(m+1)} = \hbar\sqrt{(l-m)(l+m+1)},$$
$$B_l^m = \hbar\sqrt{l(l+1) - m(m-1)} = \hbar\sqrt{(l+m)(l-m+1)}.$$

Note what happens at the top and bottom of the ladder (i.e. when you apply L_+ to f_l^l or L_- to f_l^{-l}).

Solution (4.21)

Use Eq. 4.112: $L_\mp L_\pm = L^2 - L_z^2 \mp \hbar L_z$ and work out: $\langle f_l^m | L_\mp L_\pm f_l^m \rangle$ to obtain the norms squared $\langle L_+ f_l^m | L_+ f_l^m \rangle$ and $\langle L_- f_l^m | L_- f_l^m \rangle$.

Problem 4.25

(a) What is $L_+Y_l^l$? (No calculation allowed!)
(b) Use the result of (a), together with Equation 4.130 and the fact that $L_zY_l^l = \hbar l Y_l^l$, to determine $Y_l^l(\theta, \phi)$, up to a normalization constant.
(c) Determine the normalization constant by direct integration. Compare your final answer to what you got in Problem 4.7.

Solution (4.25)

(a) $L_+Y_l^l = 0$ (why is this?)

(b) $Y_l^l(\theta, \phi) = A\left(e^{i\phi}\sin\theta\right)$

(c) $A = \frac{(-1)^l}{2^{l+1}\,l!}\sqrt{\frac{(2l+1)!}{\pi}}$.

Problem 4.26

In Problem 4.4 you showed that

$$Y_2^1(\theta, \phi) = -\sqrt{15/8\pi}\,\sin\theta\cos\theta\, e^{i\phi}.$$

Apply the raising operator to find $Y_2^2(\theta, \phi)$. Use Equation 4.121 to get the normalization.

Solution (4.26)

$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\left(e^{i\phi}\sin\theta\right)^2$.

Problem 4.30

An electron is in the spin state

$$\chi = A\begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

(a) Determine the normalization constant A.

(b) Find the expectation values of S_x, S_y, and S_z.

(c) Find the "uncertainties" σ_{S_x}, σ_{S_y}, and σ_{S_z}. Note: These sigmas are standard deviations, not Pauli matrices!

(d) Confirm that your results are consistent with all three uncertainty principles (Equation 4.100 and its cyclic permutations—only with S in place of L, of course)

Solution (4.30)

(a) $A = 1/5$.

(b) $\langle S_x \rangle = 0$, $\langle S_y \rangle = -\frac{12}{25}\hbar$, and $\langle S_z \rangle = -\frac{7}{50}\hbar$.

(c) $\sigma_{S_x} = \frac{\hbar}{2}$, $\quad \sigma_{S_y} = \frac{7}{50}\hbar$, $\quad \sigma_{S_z} = \frac{12}{25}\hbar$.

(d) Example: $\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2} \cdot \frac{7}{50}\hbar \geq \frac{\hbar}{2}|\langle S_x \rangle| = \frac{\hbar}{2} \cdot \frac{7}{50}\hbar$, i.e. right at the uncertainty limit.

Show that $\sigma_{S_y}\sigma_{S_z}$ and $\sigma_{S_z}\sigma_{S_x}$ also fullfil the uncertainty principle.

Problem 4.34

Construct the spin matrices (S_x, S_y, and S_z) for a particle of spin 1. *Hint:* How many eigenstates of S_z are there? Determine the action of S_z , S_+, and S_- on each of these states. Follow the procedure used in the text for spin 1/2.

Solution (4.34)

If we choose the three spin states as:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

Then we can write the spin matrices as:

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Problem 4.35

In Example 4.3:

(a) If you measured the component of spin angular momentum along the x direction, at time t, what is the probability that you would get $+\hbar/2$?

(b) Same question, but for the y component.

(c) Same, for the z component.

Solution (4.35)

(a) $P_+^{(x)}(t) = \frac{1}{2}\left[1 + \sin\alpha\cos(\gamma B_0 t)\right]$.
(b) $P_+^{(y)}(t) = \frac{1}{2}\left[1 - \sin\alpha\cos(\gamma B_0 t)\right]$.
(c) $P_+^{(z)}(t) = \cos^2\frac{\alpha}{2}$.

Problem 4.37

(a) Apply S_- to $|10\rangle$ (Equation 4.175), and confirm that you get $\sqrt{2}\hbar|1-1\rangle$.
(b) Apply $S_\pm$ to $|00\rangle$ (Equation 4.176), and confirm that you get zero.
(c) Show that $|11\rangle$ and $|1-1\rangle$ (Equation 4.175) are eigenstates of S^2, with the appropriate eigenvalue.

Solution (4.37)

Applying operators $S_\pm$ to a state with two spins, e.g. $|\uparrow\uparrow\rangle$, can be split in applying one operator to the first spin and another one to the second one: $S_\pm = S_\pm^{(1)} + S_\pm^{(2)}$.
Consider the line above Eq. 4.146: $S_-|\uparrow\rangle = \hbar|\downarrow\rangle$ and $S_-|\downarrow\rangle = 0$.

Problem 4.45

(a) Derive Equation 4.199 from Equation 4.190.

(b) Derive Equation 4.211, starting with Equation 4.210.

Solution (4.45)

(a) Use the identities $\nabla\cdot\vec{A} = 0$ and $\varphi = 0$ (see comments after Eq. 4.198).

(b) Apply $(-i\hbar\nabla - q\vec{A})$ to both sides of Eq. 4.210.

Problem 4.52

(a) Construct the spatial wave function (ψ) for hydrogen in the state $n = 3, l = 2, m = 1$. Express your answer as a function of r, θ, ϕ, and a (the Bohr radius) only—no other variables (ρ, z, etc.) or functions (Y, v, etc.), or constants (A, c_0, etc.), or derivatives, allowed (π is okay, and e, and 2, etc.).
(b) Check that this wave function is properly normalized, by carrying out the appropriate integrals over r, θ, and ϕ.
(c) Find the expectation value of r^s in this state. For what range of s (positive and negative) is the result finite?

Solution (4.52)

(a) $\psi_{321} = -\frac{1}{\sqrt{\pi}}\frac{1}{81a^{7/2}}\, r^2 e^{-r/3a} \sin\theta \cos\theta \, e^{i\phi}$.

(b) $\langle r^s \rangle = (s+6)!\left(\frac{3a}{2}\right)^s \frac{1}{720}$ which is finite for $s > -7$.

Problem 4.53

(a) Construct the wave function for hydrogen in the state $n = 4, l = 3, m = 3$. Express your answer as a function of the spherical coordinates r, θ, and ϕ.

(b) Find the expectation value of r in this state. (As always, look up any nontrivial integrals.)

(c) If you could somehow measure the observable $L_x^2 + L_y^2$ on an atom in this state, what value (or values) could you get, and what is the probability of each?

Solution (4.53)

(a) $\psi_{433} = -\frac{1}{6144\sqrt{\pi}a^{9/2}}\, r^3 \, e^{-r/4a} \sin^2\theta \, e^{i3\phi}$.

(b) $\langle r \rangle = 18a$.

(c) $L_x^2 + L_y^2 = 3\hbar^2$ with probability 1.

Problem 4.54

What is the probability that an electron in the ground state of hydrogen will be found inside the nucleus?

(a) First calculate the exact answer, assuming the wave function (Equation 4.80) is correct all the way down to $r = 0$. Let b be the radius of the nucleus.

(b) Expand your result as a power series in the small number $\epsilon \equiv 2b/a$, and show that the lowest-order term is the cubic: $P \approx (4/3)\,(b/a)^3$. This should be a suitable approximation, provided that $b \ll a$ (which it is).

(c) Alternatively, we might assume that $\psi(r)$ is essentially constant over the (tiny) volume of the nucleus, so that $P \approx (4/3)\pi b^3 |\psi(0)|^2$. Check that you get the same answer this way.

(d) Use $b \approx 10^{-15}$ m and $a \approx 0.5 \times 10^{-10}$ m to get a numerical estimate for P. Roughly speaking, this represents the "fraction of its time that the electron spends inside the nucleus."

Solution (4.54)

(a) $P = 1 - \left(1 + \frac{2b}{a} + 2\frac{b^2}{a^2}\right) e^{-2b/a}$.

(b) $P = \frac{4}{3}\left(\frac{b}{a}\right)^3$.

(c) $P = 1.07 \times 10^{-14}$.

Problem 4.58

An electron is in the spin state

$$\chi = A \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix}.$$

(a) Determine the constant A by normalizing χ.

(b) If you measured S_z on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_z?

(c) If you measured S_x on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_x?

(d) If you measured S_y on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_y?

Solution (4.58)

(a) $A = 1/3$.

(b) $\frac{\hbar}{2}$ with probability $\frac{5}{9}$; $-\frac{\hbar}{2}$ with probability $\frac{4}{9}$.
$\langle S_z \rangle = \frac{\hbar}{18}$.

(c) $\frac{\hbar}{2}$ with probability $\frac{13}{18}$; $-\frac{\hbar}{2}$ with probability $\frac{5}{18}$.
$\langle S_x \rangle = \frac{2\hbar}{9}$.

(d) $\frac{\hbar}{2}$ with probability $\frac{17}{18}$; $-\frac{\hbar}{2}$ with probability $\frac{1}{18}$.
$\langle S_y \rangle = \frac{4\hbar}{9}$.