PHOT 301: Quantum Photonics

Homework problems 3

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Problems

Here we list the problems with there final solutions so you can check whether you have the corrects answers. Some problems ask you to prove a theorem, for these problems, I write just some extra hints. The problems are from Griffiths 3rd edition. The problems for this week:

- Chapter 2: 2.31, 2.34, 2.41, 2.53 (please see the previous solutions file)
- Chapter 3: 3.1, 3.4, 3.7, 3.10, 3.12, 3.14, 3.19, 3.25, 3.26, 3.33, 3.44

Problem 3.1

(a) Show that the set of all square-integrable functions is a vector space (refer to Section A.1 for the definition). Hint: The main point is to show that the sum of two square-integrable functions is itself square-integrable. Use Equation 3.7. Is the set of all normalized functions a vector space?

(b) Show that the integral in Equation 3.6 satisfies the conditions for an inner product (Section A.2).

Solution (3.1)

(a) The set of all normalized functions does not form a vector space. Think about simple counter-examples: which normalized function represents the zero-vector? Is the sum of two normalized functions again a normalized function?
(b) Use the fact that integration is a linear operation.

Problem 3.4

(a) Show that the sum of two hermitian operators is hermitian.

(b) Suppose $\hat{Q}$ is hermitian, and α is a complex number. Under what condition (on α) is $\alpha\hat{Q}$ hermitian?

(c) When is the product of two hermitian operators hermitian?

(d) Show that the position operator $\hat{x}$ and the Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$ are hermitian.

Solution (3.4)

(b) When α is real.

(c) The product of two operators $\hat{A}\hat{B}$ is hermitian when they commute: $[\hat{A}, \hat{B}] = 0$.

(d) To prove that $\langle \hat{H}f|g\rangle = \langle f|\hat{H}g\rangle$, use integration by parts twice and make use of the fact that f and g are zero at $x = \pm\infty$ (the boundary integrals vanish).

Problem 3.7

(a) Suppose that $f(x)$ and $g(x)$ are two eigenfunctions of an operator $\hat{Q}$, with the same eigenvalue q. Show that any linear combination of f and g is itself an eigenfunction of $\hat{Q}$, with eigenvalue q.

(b) Check that $f(x) = \exp(x)$ and $g(x) = \exp(-x)$ are eigenfunctions of the operator d^2/dx^2, with the same eigenvalue. Construct two linear combinations of f and g that are orthogonal eigenfunctions on the interval $(-1, 1)$.

Solution (3.7)

(a) Define $h(x) = af(x) + bg(x)$ and start with $\hat{Q}h(x) = \hat{Q}(af(x) + bg(x)) = ...$

(b) For the new orthogonal eigenfunctions: Try to make even and odd linear combinations.

Problem 3.10

Is the ground state of the infinite square well an eigenfunction of momentum? If so, what is its momentum? If not, why not? [For further discussion, see Problem 3.34.]

Solution (3.10)

The ground state of the infinite square well is not an eigenfunction of the momentum operator.

Problem 3.12

Find $\Phi(p,t)$ for the free particle in terms of the function $\phi(k)$ introduced in Equation 2.101. Show that for the free particle $|\Phi(p,t)|^2$ is independent of time. Comment: the time independence of $|\Phi(p,t)|^2$ for the free particle is a manifestation of momentum conservation in this system.

Solution (3.12)

Fill in $k = p/\hbar$ in Eq. 2.101 and identify $\Phi(p,t)$ from Eq. 3.55.

Problem 3.14

(a) Prove the following commutator identities:

$$\left[\hat{A}+\hat{B},\hat{C}\right] = \left[\hat{A},\hat{C}\right] + \left[\hat{B},\hat{C}\right]$$

$$\left[\hat{A}\hat{B},\hat{C}\right] = \left[\hat{A},\hat{B}\right] C + \left[\hat{A},\hat{C}\right]\hat{B}$$

(b) Show that

$$[x^n,\hat{p}] = i\hbar n x^{n-1}$$

(c) Show more generally that

$$[f(x),\hat{p}] = i\hbar \frac{df(x)}{dx},$$

for any function $f(x)$ that admits a Taylor series expansion.

(d) Show that for the simple harmonic oscillator:

$$[\hat{H},\hat{a}_\pm] = \pm\hbar\omega\hat{a}_\pm$$

Hint: Use Equation 2.54.

Solution (3.14)

For (b) and (c) use a test function $g(x)$.

For (d): Express the Hamiltonian with ladder operators, and afterwards make use of Eq. 2.56.

Problem 3.19

Use Equation 3.73 (or Problem 3.18 (c) and (d)) to show that:

(a) For any (normalized) wave packet representing a free particle ($V(x) = 0$), $\langle x \rangle$ moves at constant velocity (this is the quantum analog to Newton's first law). Note: You showed this for a gaussian wave packet in Problem 2.42, but it is completely general.
(b) For any (normalized) wave packet representing a particle in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$, $\langle x \rangle$ oscillates at the classical frequency. Note: You showed this for a particular gaussian wave packet in Problem 2.49, but it is completely general.

Solution (3.19)

(a) From Eq. 1.38 (Ehrenfest's theorem), one can show that $\langle \hat{p} \rangle$ should be a constant. Apply then the generalized Ehrenfest's theorem in Eq. 3.73, with the position operator $\hat{Q} = \hat{x}$.

(b) In a similar manner as in (a) you can show that the expectation value for the position operator in an harmonic oscillator equals:

$$\frac{d\langle x \rangle}{dx^2} = -\omega^2 \langle x \rangle$$

with known oscillating solutions: $\langle x \rangle = A\sin(\omega t) + B\cos(\omega t)$

Problem 3.25

The Hamiltonian for a certain two-level system is

$$\hat{H} = \epsilon(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

where $|1\rangle$, $|2\rangle$ is an orthonormal basis and ϵ is a number with the dimensions of energy. Find its eigenvalues and eigenvectors (as linear combinations of $|1\rangle$ and $|2\rangle$). What is the matrix H representing $\hat{H}$ with respect to this basis?

Solution (3.25)

When using the vector representation in $\mathbb{R}^2$, we can write the orthonormal basis-vectors and the Hamiltonian matrix as follows:

$$H = \epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Write the linear combinations as $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$. Eigenenergies $E = \pm\sqrt{2}\epsilon$ and eigenstates $|\psi_{\pm}\rangle = c_1[|1\rangle + (\pm\sqrt{2} - 1)|2\rangle]$

Problem 3.26

Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle, \quad |\beta\rangle = i|1\rangle + 2|3\rangle.$$

(a) Construct $\langle\alpha|$ and $\langle\beta|$ (in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$).
(b) Find $\langle\alpha|\beta\rangle$ and $\beta|\alpha$, and confirm that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.
(c) Find all nine matrix elements of the operator $\hat{A} \equiv |\alpha\rangle\langle\beta|$, in this basis, and construct the matrix A. Is it hermitian?

Solution (3.26)

(a) $\langle\alpha| = -i\langle 1| - 2\langle 2| + i\langle 3|; \quad \langle\beta| = -i\langle 1| + 2\langle 3|.$

(b) $\langle\alpha|\beta\rangle = 1 + 2i$ and $\langle\beta|\alpha\rangle = 1 - 2i = \langle\alpha|\beta\rangle^*$
(c) The matrix A is not Hermitian and is given by:

$$\begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}$$

Problem 3.33

Sequential measurements. An operator $\hat{A}$, representing observable A, has two (normalized) eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable $\hat{B}$, has two (normalized) eigenstates φ_1 and φ_2, with eigenvalues b_1 and b_2. The eigenstates are related by

$$\psi_1 = (3\varphi_1 + 4\varphi_2)/5, \qquad \psi_2 = (4\varphi_1 - 3\varphi_2)/5.$$

(a) Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?

(b) If B is now measured, what are the possible results, and what are their probabilities?

(c) Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note that the answer would be quite different if I had told you the outcome of the B measurement.)

Solution (3.33)

(a) ψ_1

(b) b_1 with probability 9/25, and b_2 with probability 16/25.

(c) Probability to get a_1 right after the measurement of B: $\left(\frac{9}{25}\right)^2 + \left(\frac{16}{25}\right)^2 = \frac{337}{625}$.

Problem 3.44

The Hamiltonian for a certain three-level system is represented by the matrix

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$$

where a, b, and c are real numbers.

(a) If the system starts out in the state

$$|S(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

what is $|S(t)\rangle$?

(b) If the system starts out in the state

$$|S(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

what is $|S(t)\rangle$?

Solution (3.44)

(a) $|S(0)\rangle$ is an eigenvector and

$$|S(t)\rangle = e^{-ict/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

(b) $|S(0)\rangle$ is a combination of eigenvectors and

$$|S(t)\rangle = e^{-iat/\hbar}\begin{pmatrix}\cos(bt/\hbar)\\0\\-i\sin(bt/\hbar)\end{pmatrix}.$$