# PHOT 301: Quantum Photonics

**Homework problems 2**

Michaël Barbier, Summer (2024-2025)

## Problems

Here we list the problems with there final solutions so you can check whether you have the corrects answers. Some problems ask you to prove a theorem, for these problems, I write just some extra hints. The problems are from Griffiths 3rd edition. The problems for this week:

- Chapter 2: 2.11, 2.13, 2.14, 2.17, 2.18, 2.25, (2.31, 2.34, 2.41, 2.53)

Since we didn't reach the end of the Chapter this week, the last four exercises are postponed to week 3.

### Problem 2.11

(a) Compute $\langle x \rangle$, $\langle \hat{p} \rangle$, $\langle x^2 \rangle$, and $\langle \hat{p}^2 \rangle$, for the states $\psi_0$ (Equation 2.60) and $\psi_1$ (Equation 2.63), by explicit integration. Comment: In this and other problems involving the harmonic oscillator it simplifies matters if you introduce the variable $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$ and the constant $\alpha \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$.

(b) Check the uncertainty principle for these states.

(c) Compute $\langle \hat{T} \rangle$ and $\langle V \rangle$ for these states. (No new integration allowed!) Is their sum what you would expect?

#### Solution (2.11)

(a) $\langle x \rangle = 0$ and $\langle \hat{p} \rangle = 0$ for both $\psi_0$ and $\psi_1$.

$$n = 0: \qquad \langle x^2 \rangle = \frac{\hbar}{2m\omega} \qquad \langle \hat{p}^2 \rangle = \frac{m\hbar\omega}{2}$$
$$n = 1: \qquad \langle x^2 \rangle = \frac{3\hbar}{2m\omega} \qquad \langle \hat{p}^2 \rangle = \frac{3m\hbar\omega}{2}$$

(b) The eigenstates fulfill the uncertainty relation $\sigma_x \sigma_p \geq \frac{\hbar}{2}$:

$$n=0: \qquad \sigma_x\sigma_p = \frac{\hbar}{2} = \frac{\hbar}{2}$$
$$n=1: \qquad \sigma_x\sigma_p = \frac{3\hbar}{2} \geq \frac{\hbar}{2}$$

(c) The results for the kinetic and potential energy are:

$$n=0: \qquad \langle\hat{T}\rangle = \frac{1}{4}\hbar\omega \qquad \langle V\rangle = \frac{1}{4}\hbar\omega \qquad \langle\hat{T}\rangle + \langle V\rangle = \frac{1}{2}\hbar\omega = E_0$$
$$n=1: \qquad \langle\hat{T}\rangle = \frac{3}{4}\hbar\omega \qquad \langle V\rangle = \frac{3}{4}\hbar\omega \qquad \langle\hat{T}\rangle + \langle V\rangle = \frac{3}{2}\hbar\omega = E_1$$

## Problem 2.13

A Particle in the harmonic oscillator potential starts out in the state $\Psi(x,0) = A[3\psi_0(x) + 4\psi_1(x)]$.

(a) Find $A$.
(b) Construct $\Psi(x,t)$ and $|\Psi(x,t)|^2$ . Don't get too excited if $|\Psi(x,t)|^2$ oscillates at exactly the classical frequency; what would it have been had I specified $\psi_2(x)$, instead of $\psi_1(x)$?
(c) Find $\langle x\rangle$ and $\langle\hat{p}\rangle$. Check that Ehrenfest's theorem (Equation 1.38) holds, for this wave function.
(d) If you measured the energy of this particle, what values might you get, and with what probabilities?

### Solution (2.13)

(a) $A = 1/5$.

(b) $\Psi(x,t) = \frac{1}{5}[3\psi_0 e^{-i\omega t} + 4\psi_1 e^{-3i\omega t/2}]$ and $|\Psi(x,t)|^2 = \frac{1}{25}[9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1\cos(\omega t)]$.
In case one picks $\psi_2$ instead of $\psi_1$ then the frequency would double $\omega \longrightarrow 2\omega$.

(c) $\langle x\rangle = \frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)$ and $\langle\hat{p}\rangle = -\frac{24}{25}\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t)$.

(d) Possible values for the energy are:

$$E_0 = \frac{1}{2}\hbar\omega, \qquad \text{with probability} \quad |c_0|^2 = 9/25,$$
$$E_1 = \frac{3}{2}\hbar\omega, \qquad \text{with probability} \quad |c_1|^2 = 16/25,$$

## Problem 2.14

In the ground state of the harmonic oscillator, what is the probability (correct to three significant digits) of finding the particle outside the classically allowed region? Hint: Classically, the energy of an oscillator is $E = \frac{1}{2}ka^2 = \frac{1}{2}m\omega^2a^2$ , where $a$ is the amplitude. So the "classically allowed region" for an oscillator of energy $E$ extends from $-\sqrt{2E/m\omega^2}$ to $\sqrt{2E/m\omega^2}$. Look in a math table under "Normal Distribution" or "Error Function" for the numerical value of the integral, or evaluate it by computer.

### Solution (2.14)

$P_{\text{outside}} = 0.157$

## Problem 2.17

Show that $[Ae^{ikx} + Be^{-ikx}]$ and $[C \cos kx + D \sin kx]$ are equivalent ways of writing the same function of $x$, and determine the constants $C$ and $D$ in terms of $A$ and $B$, and vice versa. Comment: In quantum mechanics, when $V = 0$, the exponentials represent traveling waves, and are most convenient in discussing the free particle, whereas sines and cosines correspond to standing waves, which arise naturally in the case of the infinite square well.

### Solution (2.17)

$C = A + B; D = i(A - B)$ and $A = \frac{1}{2}(C - iD); B = \frac{1}{2}(C + iD)$

## Problem 2.18

Find the probability current, $J$ (Problem 1.14) for the free particle wave function Equation 2.95. Which direction does the probability flow?

### Solution (2.18)

$J = \frac{\hbar k}{m}|A|^2$ which is in the positive x-direction.

## Problem 2.25

Check that the bound state of the delta-function well (Equation 2.132) is orthogonal to the scattering states (Equations 2.134 and 2.135)

**Solution (2.25)**

For wave functions $f(x)$ and $g(x)$ to be orthogonal the integral $\int_{-\infty}^{+\infty} f^*(x)g(x)\,dx$ becomes 0. We can write the integral as follows:

$$\frac{\sqrt{m\alpha}}{\hbar}\left[\int_{-\infty}^{0} e^{m\alpha x}(Ae^{ikx}+Be^{-ikx})\,dx+\int_{0}^{+\infty} e^{-m\alpha x}(Fe^{ikx}+Ge^{-ikx})\,dx\right]$$

Then by performing the integral and applying boundary conditions for the relations between $A$, $B$, $F$, and $G$, the integral can shown to be zero.

## Problem 2.31

The Dirac delta function can be thought of as the limiting case of a rectangle of area 1, as the height goes to infinity and the width goes to zero. Show that the delta- function well (Equation 2.117) is a "weak" potential (even though it is infinitely deep), in the sense that $z_0 \longrightarrow 0$. Determine the bound state energy for the delta-function potential, by treating it as the limit of a finite square well. Check that your answer is consistent with Equation 2.132. Also show that Equation 2.172 reduces to Equation 2.144 in the appropriate limit.

**Solution (2.31)**

Start with the formula for $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$ for the finite well, then take the limit for the width of the well $2a$ going to zero while the area $2aV_0$ is constant. Show that this limit is zero.

Afterwards, notice that any "crossing" in the graphical representation of the trancendental equation is at small $z$ because $z_0$ is small. For small $z$ you can simplify the trancendental equation resulting in energy $E = -\frac{m\alpha^2}{2\hbar^2}$ similar to the delta-function potential well.

## Problem 2.34

Consider the "step" potential:

$$V(x) = \{0, x \leq 0, V_0, x > 0.$$

(a) Calculate the reflection coefficient, for the case $E < V0$, and comment on the answer.

(b) Calculate the reflection coefficient for the case $E > V0$.

(c) For a potential (such as this one) that does not go back to zero to the right of the barrier, the transmission coefficient is not simply $|F|^2/|A|^2$ (with $A$ the incident amplitude and $F$ the transmitted amplitude), because the transmitted wave travels at a different speed. Show that:

$$T = \sqrt{\frac{E - V_0}{E}} \frac{|F|^2}{|A|^2},$$

for $E > V_0$. Hint: You can figure it out using Equation 2.99, or—more elegantly, but less informatively—from the probability current (Problem 2.18). What is $T$, for $E < V_0$?
(d) For $E > V_0$, calculate the transmission coefficient for the step potential, and check that $T + R = 1$.

### Solution (2.34)

Assume that the wave function before the step $\psi(x \leq 0) = Ae^{ikx} + Be^{-ikx}$ and after the step $\psi(x \geq 0) = Fe^{ikx}$.

(a) For energy $E < V_0$: $R = |B|^2/|A|^2 = 1$

(b) For energy $E > V_0$: $R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}$

(c) Consider that the phase velocity of the incident wave $v_i$ is different from the one of the transmitted wave $v_t$, after the step (see Eq. 2.98), and the probability to transmit is therefore also proportional to the ratio $v_t/v_i$.
Alternatively use the probability current, see problem 2.18.

(d) Use the expression of (c) and apply continuity of $\psi(x)$ and its derivative $\psi'(x)$ to extract:

$$T = \frac{4\sqrt{E}\sqrt{E - V_0}(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2}$$

Then use the above $T$, and $R$ from (b) to show that $T + R = 1$.

## Problem 2.41

Find the allowed energies of the half harmonic oscillator

$$V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2, & x > 0, \\ \infty, & x < 0. \end{cases}$$

(This represents, for example, a spring that can be stretched, but not compressed.) Hint: This requires some careful thought, but very little actual calculation.

**Solution (2.41)**

The allowed energies are corresponding to odd solutions:

$$E_n = (n + 1/2)\hbar\omega, \qquad n = 1, 3, 5, ...$$

## Problem 2.53

The Scattering Matrix. The theory of scattering generalizes in a pretty obvious way to arbitrary localized potentials (Figure 2.21). To the left (Region I), $V(x) = 0$, so:

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

To the right (Region III), V (x) is again zero, so

$$\psi(x) = Fe^{ikx} + Ge^{-ikx}$$

In between (Region II), of course, I can't tell you what $\psi$ is until you specify the potential, but because the Schrödinger equation is a linear, second-order differential equation, the general solution has got to be of the form

$$\psi(x) = Cf(x) + Dg(x),$$

where $f(x)$ and $g(x)$ are two linearly independent particular solutions. There will be four boundary conditions (two joining Regions I and II, and two joining Regions II and III). Two of these can be used to eliminate $C$ and $D$, and the other two can be "solved" for $B$ and $F$ in terms of $A$ and $G$:

$$B = S_{11}A + S_{12}G, \quad F = S_{21}A + S_{22}G.$$

The four coefficients $S_{ij}$, which depend on $k$ (and hence on $E$), constitute a $2\times 2$ matrix $S$, called the scattering matrix (or $S$-matrix, for short). The $S$-matrix tells you the outgoing amplitudes ($B$ and $F$) in terms of the incoming amplitudes ($A$ and $G$):

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$

In the typical case of scattering from the left, $G = 0$, so the reflection and transmission coefficients are

$$R_l = \frac{|B|^2}{|A|^2}\bigg|_{G=0} = |S_{11}|^2, \qquad T_l = \frac{|F|^2}{|A|^2}\bigg|_{G=0} = |S_{21}|^2,$$

For scattering from the right, $A = 0$, and

$$R_r = \frac{|F|^2}{|G|^2}\bigg|_{A=0} = |S_{22}|^2, \qquad T_r = \frac{|B|^2}{|G|^2}\bigg|_{A=0} = |S_{12}|^2,$$

(a) Construct the S-matrix for scattering from a delta-function well (Equation 2.117).
(b) Construct the S-matrix for the finite square well (Equation 2.148). Hint: This requires no new work, if you carefully exploit the symmetry of the problem

**Solution (2.53)**

(a) S-matrix for scattering from a delta-function well:

$$S = \frac{1}{1 - i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}.$$

(b) S-matrix for the finite square well:

$$S = \frac{e^{-2ika}}{\cos(2la) - i\frac{k^2+l^2}{2kl}\sin(2la)}\begin{pmatrix} i\frac{l^2-k^2}{2kl}\sin(2la) & 1 \\ 1 & i\frac{l^2-k^2}{2kl}\sin(2la) \end{pmatrix}.$$