PHOT 301: Quantum Photonics

Homework problems 1

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Problems

Here we list the problems with there final solutions so you can check whether you have the corrects answers. Some problems ask you to prove a theorem, for these problems, I write just some extra hints. The problems are from Griffiths 3rd edition. The problems for this week:

- Chapter 1: 1.1, 1.2, 1.3, 1.5, 1.8
- Chapter 2: 2.1(c), 2.3, 2.4, 2.5, 2.7

Problem 1.1

For the distribution of ages in the example in Section 1.3.1:

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$.
(b) Determine Δj for each j, and use Equation 1.11 to compute the standard deviation.
(c) Use your results in (a) and (b) to check Equation 1.12.

Solution (1.1)

(a) $\langle j^2 \rangle = 459.6$ and $\langle j \rangle^2 = 441$.
(b) Standard deviation $\sigma = 4.3$.
(c) $4.3 = \langle (\Delta j)^2 \rangle = \sqrt{18.6}$?

Problem 1.2

(a) Find the standard deviation of the distribution in Example 1.2.
(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from the average?

Solution (1.2)

(a) Standard deviation $\sigma = \frac{2}{3\sqrt{5}}h$.
(b) $1 - \sqrt{\frac{1}{3}(1+2/\sqrt{5})} + \sqrt{\frac{1}{3}(1-2/\sqrt{5})}$.

Problem 1.3

Consider the gaussian distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are positive real constants. (The necessary integrals are inside the back cover.)

(a) Use Equation 1.16 to determine A.
(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.
(c) Sketch the graph of $\rho(x)$.

Solution (1.3)

(a) $A = \sqrt{\lambda/\pi}$.
(b) $\langle x \rangle = a$, $\langle x^2 \rangle = \frac{1}{2\lambda} + a^2$, and $\sigma = \frac{1}{\sqrt{2\lambda}}$.
(c) Bell curve at $x = a$ and maximum $\rho(a) = A$.

Problem 1.5

Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t},$$

where A, λ, and ω are positive real constants. (We'll see in Chapter 2 for what potential (V) this wave function satisfies the Schrödinger equation.) (a) Normalize Ψ. (b) Determine the expectation values of x and x^2 . (c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x \rangle + \sigma)$ and $(\langle x \rangle - \sigma)$, to illustrate the sense in which σ represents the "spread" in x. What is the robability that the particle would be found outside this range?

Solution (1.5)

(a) $A = \sqrt{\lambda}$.
(b) $\langle x \rangle = 0$, $\langle x^2 \rangle = \frac{1}{2\lambda^2}$.
(c) $\sigma = \frac{1}{\sqrt{2}\lambda}$. $P(|x| > \sigma) = e^{-\lambda}$.

Problem 1.8

Suppose you add a constant V_0 to the potential energy (by "constant" I mean independent of x as well as t). In classical mechanics this doesn't change anything, but what about quantum mechanics? Show that the wave function picks up a time-dependent phase factor: $\exp(-iV_0 t/\hbar)$. What effect does this have on the expectation value of a dynamical variable?

Solution (1.8)

No effect.

Problem 2.1(c)

Prove theorem (c): If $V(x)$ is an even function (that is, $V(-x) = V(x)$) then $\psi(x)$ can always be taken to be either even or odd. Hint: If $\psi(x)$ satisfies Equation 2.5, for a given E, so too does $\psi(-x)$, and hence also the even and odd linear combinations $\psi(x) \pm \psi(-x)$.

Partial solution (2.1c)

Prove via considering $\psi_\pm(x) = \psi(x) \pm \psi(-x)$ and ψ can always be written as a linear combination of them $\psi_\pm(x)$.

Problem 2.3

Show that there is no acceptable solution to the (time-independent) Schrödinger equation for the infinite square well with $E = 0$ or $E < 0$. (This is a special case of the general theorem in Problem 2.2, but this time do it by explicitly solving the Schrödinger equation, and showing that you cannot satisfy the boundary conditions.)

Solution (2.3)

- For $E = 0$ the solution is $\psi(x) = Ax + B$, prove then that A and B are zero due to the boundary conditions $\psi(0) = \psi(a) = 0$.
- For $E < 0$ the solution can be written as $\psi(x) = A\sinh(\kappa x) + B\cosh(\kappa x)$ (alternatively use $\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$). Prove then that A and B are zero due to the boundary conditions $\psi(0) = \psi(a) = 0$.

Problem 2.4

Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, σ_x, and σ_p, for the nth stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closest to the uncertainty limit?

Solution (2.4)

$$\langle x\rangle = \frac{a}{2}, \qquad \langle x^2\rangle = a^2\left(\frac{1}{3}-\frac{1}{2\pi^2 n^2}\right),$$

$$\langle p\rangle = 0, \qquad \langle p^2\rangle = 2mE_n = \frac{\hbar^2\pi^2 n^2}{a^2},$$

$$\sigma_x = a\sqrt{\frac{1}{12}-\frac{1}{2\pi^2 n^2}}, \qquad \sigma_p = \frac{\hbar\pi n}{a}$$

$$\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{\pi^2 n^2}{3}-2} > \frac{\hbar}{2}, \qquad \text{and smallest when in ground state } n=1$$

Problem 2.5

A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x,0) = A[\psi_1(x)+\psi_2(x)].$$

(a) Normalize $\Psi(x,0)$. (That is, find A. This is very easy, if you exploit the orthonormality of $\psi_1(x)$ and $\psi_2(x)$. Recall that, having normalized Ψ at $t=0$, you can rest assured that it stays normalized—if you doubt this, check it explicitly after doing part (b).)

(b) Find $\Psi(x,t)$ and $|\Psi(x,t)|2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega \equiv \frac{\pi^2\hbar}{2ma^2}$.

(c) Compute $\langle x\rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation? (If your amplitude is greater than $a/2$, go directly to jail.)

(d) Compute $\langle p\rangle$. (As Peter Lorre would say, "Do it ze kveek vay, Johnny!")

(e) If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of $\mathcal{H}$. How does it compare with E_1 and E_2?

Solution (2.5)

(a) $A = \frac{1}{\sqrt{2}}$.

(b) $\Psi(x,t) = \frac{1}{\sqrt{a}}\, e^{-i\omega t}\left[\sin\frac{\pi x}{a} + \sin\frac{\pi x}{a}\, e^{-i3\omega t}\right]$.
$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right)+\sin^2\left(\frac{2\pi x}{a}\right)+2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right]$.

(c) $\langle x\rangle = \frac{a}{2}\left[1-\frac{32}{9\pi^2}\cos(3\omega t)\right]$, where the Amplitude is $\frac{a}{2}\frac{32}{9\pi^2} < \frac{a}{2}$ and the angular frequency $3\omega = 3E_1/\hbar = \frac{3\hbar\pi^2}{2ma^2}$

(d) $\langle p\rangle = \frac{8\hbar}{3a}\sin(3\omega t)$.

(e) $\langle\hat{H}\rangle = \frac{5}{2}E_1 = \frac{5}{2}\frac{\hbar^2\pi^2}{2ma^2}$

Problem 2.7

A particle in the infinite square well has the initial wave function

$$\Psi(x,0) = \begin{cases} Ax, & 0 \le x \le a/2, \\ A(a-x), & a/2 \le x \le a. \end{cases}$$

(a) Sketch $\Psi(x,0)$, and determine the constant A.
(b) Find $\Psi(x,t)$.
(c) What is the probability that a measurement of the energy would yield the value E_1 ?
(d) Find the expectation value of the energy, using Equation 2.21.

Solution (2.7)

(a) The sketch is a isosceles triangle with base from $x = 0$ to $x = a$ and top $Aa/2$. Normalization constant $A = \frac{2\sqrt{3}}{\sqrt{a^3}}$
(b) $\Psi(x,t) = \sum_n c_n \psi_n(x)\, e^{-in^2 E_1 t/\hbar}$ with $c_n = \frac{4\sqrt{6}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$. Notice that $c_n \neq 0$ only when $n = 1, 3, 5, \ldots$. Therefore:

$$\Psi(x,t) = \sqrt{\frac{2}{a}}\frac{4\sqrt{6}}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2}(-1)^{(n-1)/2} \sin\left(\frac{n\pi x}{a}\right) e^{-in^2 E_1 t/\hbar}$$

(c) $P(E_1) = |c_1|^2 = \frac{16\cdot 6}{\pi^4}$
(d) $\langle \hat{H} \rangle = \frac{6\hbar^2}{ma^2}$.