PHOT 301: Quantum Photonics

Homework: Solving Systems of Equations

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A system of equations

Examples of systems of equations:

$$\begin{cases} 4x - y + 3z = 5 \\ 12x + y + z = -2 \\ 2x - 1z = 1 \end{cases} \qquad \begin{cases} 2x^2 - x + 2 = 0 \\ 3x^2 + x = 0 \\ 2x - 1 = 0 \end{cases} \qquad \begin{cases} \sin(x) + 5\cos(x) = 3 \\ 2\sin(x) - 2\cos(x) = 1 \end{cases}$$

Where you can imagine that we have independent variables x, y, z, or polynomials in x or linear independent functions $\sin(x)$, $\cos(x)$. We can convert each of these systems of equations in independent variables to matrix equations:

$$\begin{pmatrix} 4 & -1 & 3 \\ 12 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In general we can write a system of equations with coefficients a_{ij} for the i^{th} equation and j corresponding to the variable x_j. The values on the right-hand-side are denoted b_i. This system of equations corresponds then to a matrix system $A\vec{x} = \vec{b}$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Solving a system of equations

Suppose a simple example system in two variables:

$$\begin{cases} 2x - y = 1 \\ 3x + y = -2 \end{cases} \qquad \text{Solutions:} \quad x = -\frac{1}{5}, \quad y = -\frac{7}{5}$$

There are multiple ways to solve a system of equations:

Solving via substitution:

$$\begin{cases} 2x - y = 1 \\ 3x + y = -2 \end{cases} \Rightarrow \begin{cases} y = 2x - 1 \\ 3x + (2x-1) = -2 \end{cases} \Rightarrow \begin{cases} y = 2x - 1 \\ 5x = -1 \end{cases} \Rightarrow \begin{cases} y = -\frac{2}{5} - 1 \\ x = -\frac{1}{5} \end{cases} \Rightarrow \begin{cases} y = -\frac{7}{5} \\ x = -\frac{1}{5} \end{cases}$$

which gives the solutions above. We can also solve the system by using the matrix notation:

$$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Solving using the inverse matrix:

$$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -1 \\ -7 \end{pmatrix}$$

Where we used the definition of the inverse of a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - cd} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Solving via Cramer's rule:

Cramer's rule uses the definition of the general inverse to simplify and directly solve the system (we color-coded the columns to better understand what is going on):

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \Rightarrow \qquad x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

The extension to larger systems is similar:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Rightarrow \qquad x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{13} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\det[A]}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\det[A]} \qquad z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\det[A]}$$

where

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solving via Gaussian elimination

Gaussian elimination is based on applying three elementary row operations:

- Swapping two rows (e.g. $R_1 \leftrightarrow R_2$),
- Multiplying a row by a nonzero number (e.g. $2R_3$),
- Adding a multiple of one row to another row (e.g. $R_2 - 3\,R_1$).

Using these operations one can bring a matrix always in the form of a **upper triangular matrix** (echelon form) and from there in a **reduced row echelon form** where the leading nonzero elements are equal to one. Let's take the following system of equations as example:

$$\begin{pmatrix} 4 & -1 & 3 \\ 12 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix},$$

To solve it we first bring it in the **augmented matrix** form, basically we add the right-hand-side $\vec{b}$ as an extra column:

$$\left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 12 & 1 & 1 & -2 \\ 2 & 0 & -1 & 1 \end{array}\right]$$

Then we use the above three rules to transform the matrix into an upper triangular matrix. We start with the first column:

$$\left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 12 & 1 & 1 & -2 \\ 2 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_2 - 3\,R_1} \left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 4 & -8 & -17 \\ 2 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}\,R_1} \left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 4 & -8 & -17 \\ 0 & 1/2 & -5/2 & -3/2 \end{array}\right]$$

Then we continue with the second column:

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 1/2 & -5/2 & -3/2 \\ 0 & 4 & -8 & -17 \end{array}\right] \xrightarrow{R_3 - 8\,R_2} \left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 1/2 & -5/2 & -3/2 \\ 0 & 0 & 2 & -5 \end{array}\right]$$

Ending up with a upper triangular matrix form. This could now be solved by substitution, but we can also continue to the reduced row echelon form:

$$\xrightarrow{\frac{1}{4}\,R_1} \left[\begin{array}{ccc|c} 1 & -1/4 & 3/4 & 5/4 \\ 0 & 1/2 & -5/2 & -3/2 \\ 0 & 0 & 2 & -5 \end{array}\right] \xrightarrow{2\,R_2} \left[\begin{array}{ccc|c} 1 & -1/4 & 3/4 & 5/4 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 2 & -5 \end{array}\right] \xrightarrow{\frac{1}{2}\,R_3} \left[\begin{array}{ccc|c} 1 & -1/4 & 3/4 & 5/4 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & -5/2 \end{array}\right]$$

$$\xrightarrow{R_1 + \frac{1}{4}\,R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & -5/2 \end{array}\right] \xrightarrow{R_2 + 5\,R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -5/2 \end{array}\right] \xrightarrow{R_1 + \frac{1}{2}\,R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -5/2 \end{array}\right]$$

Resulting in the solution: $x = -3/4$, $y = -14$, and $z = -5/2$.

Obtaining the inverse matrix via Gaussian elimination

Above we saw that applying the appropriate sequence of elementary row operations we can reach to the unit matrix for the first 3 columns (representing matrix A in $Ax = b$). Thereby b was transformed along and we obtained the values for the solution $x = -3/4$, $y = -14$, and $z = -5/2$.

If we put now instead of b the unit matrix $\mathbb{1}$ then we can find the inverse matrix. The idea is that we apply the same sequence of elementary matrices E_j to both A and $\mathbb{1}$ in the equation $Ax = \mathbb{1}b$:

$$\begin{aligned} (E_N \dots E_1)A\ x &= (E_N \dots E_1)\mathbb{1}\ b \\ \Rightarrow \qquad \mathbb{1}\ x &= (E_N \dots E_1)\ b \\ \Rightarrow \qquad A^{-1}A\ x &= A^{-1}\mathbb{1}\ b \end{aligned}$$

We can use the following augmented matrix to perform the elementary row operations on both A and the unit matrix:

$$\left[\begin{array}{ccc|ccc} 4 & -1 & 3 & 1 & 0 & 0 \\ 12 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$