PHOT 301: Quantum Photonics

Homework: Matrices, inner- and outer products, and determinants

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Matrix multiplication

A matrix product is defined multiplying the row elements of the first matrix with the column elements of the second and then summing over them. For the resulting element c_{12} at row $i = 1$ and column $k = 2$ we colored the elements blue.

$$C = A \cdot B = \sum_j a_{ij} b_{jk} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Exercises on matrix multiplication:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} = \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \qquad \begin{pmatrix} 5 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

Exercises on matrix multiplication with matrices containing complex numbers:

$$\begin{pmatrix}1 & i\\ 0 & i\end{pmatrix}\begin{pmatrix}-i & 1\\ 1 & 1\end{pmatrix} = \qquad \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}2i\\ 3\end{pmatrix} =$$

$$\begin{pmatrix}0 & 0\\ -i & -1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix} = \qquad \begin{pmatrix}1 & i\\ -i & -1\end{pmatrix}\begin{pmatrix}-i\\ 1\end{pmatrix} =$$

$$\begin{pmatrix}i & -2i\\ i & -i\end{pmatrix}\begin{pmatrix}i & 2\\ 1 & i+4\end{pmatrix} = \qquad \begin{pmatrix}0 & i & 0\\ i & 0 & i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}i-1\\ 1+i\\ 2\end{pmatrix} =$$

$$\begin{pmatrix}i-1 & -1\\ -1 & i+1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \qquad \begin{pmatrix}1 & i & 0\\ -i & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}-i\\ 2+i\\ 3-i\end{pmatrix} =$$

Matrix addition

Adding matrices is simple done by element-wise addition:

$$C = A + B = a_{ij} + b_{ij} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} + \begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} = \begin{pmatrix}a_{11}+b_{11} & a_{12}+b_{12}\\ a_{21}+b_{21} & a_{22}+b_{22}\end{pmatrix}$$

Beware that multiplication has priority over matrix addition just as with numbers: $A + B \cdot C = A + (B \cdot C)$. Be clear when placing brackets to change the priority (here we used for example square brackets to distinguish them from the brackets of the matrix itself).

Exercises on matrix addition and multiplication:

$$\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\right]\begin{pmatrix}1-i\sqrt{2}\\ 1+i\sqrt{2}\end{pmatrix} = \qquad \begin{pmatrix}i\\ -1\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1 & 0 & -1\\ 1 & 1 & 0\end{pmatrix}\left[\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}0\\ i\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ -i\end{pmatrix}\right] = \qquad \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} - 14i =$$

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} + 3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \qquad \begin{pmatrix}1\\ 2\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} - 3i\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} =$$

Matrix properties and operations

The **main diagonal** of a matrix contains the elements with equal row and column vector: a_{ii}. These are diagonal elements going from top-left to bottom-right corner (if the matrix is square).

$$a_{ii} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The unit matrix: the unity matrix is a square matrix with elements on the main diagonal equal to one and all other elements zero:

$$\mathbb{1} = \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposing a matrix is like swapping row and column indices (or flipping the matrix elements around the main diagonal):

$$A^T = a_{ji} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

Complex conjugate of a matrix is the matrix with each of its elements to being the complex conjugate:

$$A^* = a^*_{ij} = \begin{pmatrix} a^*_{11} & a^*_{12} \\ a^*_{21} & a^*_{22} \end{pmatrix}$$

Hermitian adjoint of a matrix is the combination of the complex conjugate and transposed matrix:

$$A^\dagger = \left(A^T\right)^* = a^*_{ji} = \begin{pmatrix} a^*_{11} & a^*_{12} \\ a^*_{21} & a^*_{22} \end{pmatrix}^T = \begin{pmatrix} a^*_{11} & a^*_{21} \\ a^*_{12} & a^*_{22} \end{pmatrix}$$

Exercises on matrix properties:

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^T + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\right]^T = \qquad \begin{pmatrix} i \\ -i \end{pmatrix}^\dagger \begin{pmatrix} 0 & 1 \end{pmatrix}^\dagger + 1 + i =$$

$$\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}^T = \qquad \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}^\dagger \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$$

The transpose and Hermitian adjoint of a matrix have following properties:

$$[A+B]^T = A^T + B^T, \qquad [A \cdot B]^T = B^T \cdot A^T, \qquad A^T A \text{ is square and symmetric}$$
$$[A+B]^\dagger = A^\dagger + B^\dagger, \qquad [A \cdot B]^\dagger = B^\dagger \cdot A^\dagger, \qquad A^\dagger A \text{ is square and Hermitian}$$

The **inverse** of a matrix gives the unity matrix when multiplied with that matrix (we just define the inverse for a 2×2 matrix here):

$$A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{1}$$

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

A **unitary** matrix is has the property that its Hermitian adjoint is its inverse:

$$A^\dagger = A^{-1}, \qquad A^\dagger \cdot A = A \cdot A^\dagger = \mathbb{1}$$

Exercises on unitary matrices, Show that the following matrices are either unitary or not:

$$\begin{pmatrix} 0 & -i \\ i & 1 \end{pmatrix} \longrightarrow \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \qquad \frac{1}{2}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \longrightarrow \qquad \frac{1}{2}\begin{pmatrix} i & i \\ -i & i \end{pmatrix} \longrightarrow \qquad \frac{1}{4}\begin{pmatrix} 1+i & 1-i \\ 1+i & 1-i \end{pmatrix} \longrightarrow$$

Determinant

An important characteristic of matrices is the determinant of a matrix:

- The inverse of a matrix exists $\Leftrightarrow$ the determinant is nonzero
- System of equations can be solved $\Leftrightarrow$ the determinant is nonzero

The determinant of a square matrix is defined for the 2×2 matrix as follows:

$$\det(A) = \det\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Larger $N \times N$ square matrices can be calculated from Laplace's determinant expansion:

$$\det(A) = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13}$$

$$= (a_{22}a_{33} - a_{23}a_{32})a_{11} - \ldots$$

Exercises on determinants:

$$\begin{vmatrix} 1 & 2 \\ i & 0 \end{vmatrix} = \qquad \begin{vmatrix} i & 0 \\ 0 & i \end{vmatrix} = \qquad \begin{vmatrix} i & -1 \\ 1 & i \end{vmatrix} =$$

$$\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = \qquad \begin{vmatrix} 1-2i & i \\ 1 & i \end{vmatrix} = \qquad \begin{vmatrix} 2 & -2 \\ 5 & 5 \end{vmatrix} =$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \qquad \begin{vmatrix} i & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1+i & 1 \end{vmatrix} = \qquad \begin{vmatrix} -i & -2 & 1 & 1 \\ 0 & i & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{vmatrix} =$$

Hint: For some of the above determinants you can make use of below properties of determinants:

Effect of elementary transformations on the determinant:

- swapping two rows or two columns: $\det A \to -\det A$,
- Adding a multiple of a row to another row: $\det A \to \det A$ does not change.,
- Multiplying a row by a scalar c: $\det A \to c\,\det A$.

For larger matrices make also use of the fact that the determinant of triangular matrices is the product of the elements on the main diagonal:

$$\text{If } A \text{ is triangular} \qquad \longrightarrow \qquad \det[A] = \prod_i a_{ii}$$

Determinant of a matrix product is the product of the determinants:

$$\det[A\,B] = \det[A]\,\det[B]$$

The determinant does not change if you transpose the matrix: $\det[A] = \det[A^T]$

The determinant of an inverse matrix: $\det[A^{-1}] = \frac{1}{\det[A]}$

General matrix inverse

There is a general rule to compute the inverse of a matrix. In practical situations **we will not use this rule often**, since it gives rise to rather lengthy calculations. The formula is:

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$$

where the determinant should not be zero and $\mathrm{adj}(A)$ is the **adjunct** matrix. The following procedure can be used to calculate the **adjunct** matrix and inverse matrix:

1. Calculate the matrix of **minors** $M = m_{ij}$,
2. then derive the matrix of **cofactors**: $C = (-1)^{i+j}\, m_{ij}$,
3. transpose to get adjunct matrix $\mathrm{adj}(A)$, and
4. multiply by $1/\det(A)$

To illustrate this procedure we take an example 3×3 matrix:

$$A = \begin{pmatrix} 2 & 0 & 3 \\ -4 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Calculate the **minor** of every element where the **minor** of a matrix element is defined by removing the row and column of the element from the matrix and calculating the determinant of the "leftover" matrix, for example for the element on the third row and second column we obtain:

$$m_{32} = \begin{vmatrix} 2 & 0 & 3 \\ -4 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix} = 2 \cdot 0 - (-4) \cdot 3 = 12$$

The total matrix of minors becomes:

$$M = \begin{pmatrix} 0 \cdot 1 - 0 \cdot 2 & -4 \cdot 1 - 0 \cdot 0 & -4 \cdot 2 - 0 \cdot 0 \\ 0 \cdot 1 - 3 \cdot 2 & 2 \cdot 1 - 3 \cdot 0 & 2 \cdot 2 - 0 \cdot 0 \\ 0 \cdot 0 - 3 \cdot 0 & 2 \cdot 0 - 3 \cdot (-4) & 2 \cdot 0 - 0 \cdot (-4) \end{pmatrix} = \begin{pmatrix} 0 & -4 & -8 \\ -6 & 2 & 4 \\ 0 & 12 & 0 \end{pmatrix}$$

A **cofactor** of a matrix element a_{ij} is plus or minus the minor m_{ij} of that matrix element:

$$c_{32} = (-1)^{(3+2)}\, m_{32} = (-1) \cdot 12 = -12$$

This leads to a checkerboard of plus'es and minus'es

$$C = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \begin{pmatrix} 0 & -4 & -8 \\ -6 & 2 & 4 \\ 0 & 12 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & -8 \\ 6 & 2 & -4 \\ 0 & -12 & 0 \end{pmatrix}$$

Lastly, we transpose and divide by the determinant $\det(A) = -24$:

$$A^{-1} = \frac{1}{-24} \begin{pmatrix} 0 & 6 & 0 \\ 4 & 2 & -12 \\ -8 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/4 & 0 \\ -1/6 & -1/12 & 1/2 \\ 1/3 & 1/6 & 0 \end{pmatrix}$$