PHOT 301: Quantum Photonics

Homework: Eigenvalue Equations with Matrices & Operators

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Eigenvalue equations with "small" matrices

An eigenvalue equation in matrix formalism has the form:

$$A\vec{x} = \lambda\vec{x},$$

where A is a square $N \times N$ matrix (linear operator), the eigenvector $\vec{x}$ a $N \times 1$ column vector (**not the null vector**), and the eigenvalue λ a complex number.

Solving an eigenvalue equation means obtaining the eigenvalues λ_n and corresponding eigenvectors $\vec{x}_n$. For observables the operators and corresponding matrices should be Hermitian, i.e. $A^\dagger = A$, and its eigenvalues are real. There are frequently used operators however, that are not Hermitian, for example: ladder operators $\hat{a}_-$ and $\hat{a}_-^\dagger = \hat{a}_+$ of the Harmonic oscillator.

Below we sketch the procedure to solve eigenvalue equations.

STEP 1: Solving for the eigenvalues

Eigenvector $\mathbf{x} \neq \mathbf{0}$ and eigenvalues λ of matrix A:

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad (\lambda\mathbb{1} - A)\mathbf{x} = \mathbf{0}$$

Because $\mathbf{x} \neq \mathbf{0}$ the inverse of $\lambda\mathbb{1} - A$ cannot exist, meaning that its determinant should be zero:

$$\det(\lambda\mathbb{1} - A) = 0$$

Some people prefer writing $\det(A - \lambda\mathbb{1}) = 0$ which results in the same:

$$\lambda\mathbb{1} - A\vec{x} = \vec{0} = (A - \lambda\mathbb{1})\vec{x}.$$

The determinant is named the *characteristic polynomial* in λ, and the equation is called the *characteristic equation*. Solving this equation for λ gives the eigenvalues λ_n.

The highest order of λ is the dimension N of the $N \times N$ matrix. Therefore we will obtain maximum N solutions for $\lambda \longrightarrow \lambda_1, \ldots, \lambda_N$. An eigenvalue can be degenerate, meaning that it has multiple independent eigenvectors (these are also called degenerate eigenvectors then).

STEP 2: Corresponding eigenvectors

To extract an eigenvector of an eigenvalue, we fill in the specific eigenvalue in the equation and solve for vector $\vec{x}$.

$$(\lambda_n - A)\vec{x} = \lambda_n \vec{x}_n$$

The eigenvector $\vec{x}$ is only defined upon a constant factor c (normalization of the eigenvector can further specify it, but only up to a phase factor if in a complex vector space).

Example eigenvalue problem:

To illustrate the procedure of solving for the eigenvalue equation, let's take the following example equation:

$$A\vec{x} = \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \vec{x}$$

This gives for the characteristic equation: $\det(\lambda \mathbb{1} - A) = 0$:

$$\det \left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \right] = 0$$

$$\Longrightarrow \det \left[\begin{pmatrix} \lambda + 5 & -2 \\ 7 & \lambda - 4 \end{pmatrix} \right] = 0$$

The determinant is:

$$\lambda^2 + \lambda - 6 = 0 \longrightarrow (\lambda - 2)(\lambda + 3) = 0$$

Resulting in eigenvalues 2 and -3, which we could easily see by eye. For more difficult cases, remember that you can solve any quadratic equation:

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $x_\pm$ can become complex and both roots can be the same leading to degeneracy. For higher orders, finding root might be more challenging.

Now we have the eigenvalues we can extract the corresponding eigenvectors by filling in a specific eigenvalue λ_n in the original equation.

$$A\mathbf{x} = \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \qquad \lambda_1 = 2, \quad \lambda_2 = -3$$

Eigenvector $\mathbf{x}_1 = (x, y)$ for $\lambda_1 = 2$

$$A = \begin{pmatrix} \lambda_1 + 5 & -2 \\ 7 & \lambda_1 - 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad \Longrightarrow \qquad \mathbf{x}_1 = c \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

The constant c can be defined (upon a phase factor $e^{i\phi}$) by normalizing the eigenvector by $\|\mathbf{x}_n\| = \sqrt{|x|^2 + |y|^2} = 1$. This results in

$$\mathbf{x}_1 = \frac{1}{\sqrt{53}} \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

There is a special case when 2 or more eigenvalues are equal (degeneracy). In that case you can choose normalized basisvectors within the subspace.

Eigenvalue problems: large matrices

- Inverse exists ⇔ determinant is nonzero
- Determinants of 3×3 or higher order matrices A:

$$\det(A) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13}$$

$$= (a_{22}a_{33} - a_{23}a_{32})a_{11} - \ldots$$

Characteristic polynomial in λ of order N for $N \times N$ matrix. Finding roots for such higher order polynomial equations can be difficult.

Eigenvalue problems: simplify

In case we can transform the matrix to an upper or lower triangle matrix then the eigenvalues become the elements on the diagonal. We use following procedure:

1. Reduce matrix A to simpler (triangle) matrix B
2. Therefore transform matrix A by invertible matrix T (thereby the eigenvalues don't change):

$$B = T^{-1}AT \qquad \Longrightarrow \qquad \{\lambda_i\} \quad \text{the same}$$

3. The characteristic equation of upper (or lower) triangle matrices B:

$$(\lambda - b_{11})(\lambda - b_{22}) \;...\; (\lambda - b_{nn}) = 0$$

4. Derive eigenvalues and eigenvectors for B:

$$\Longrightarrow \quad \begin{cases} \text{Eigenvalues} & \lambda_i = b_{ii} \\ \text{Eigenvectors} & \mathbf{x'}_i \text{ of } B = T\mathbf{x}_i \end{cases}$$

Therefore, it is always good to try to convert the matrix to an upper or lower triangle matrix to obtain the eigenvalues. Just remember that you need to go to the original matrix if you want to extract the eigenvectors.

Exercises on eigenvalue problems:

Solve the following eigenvalue equations for eigenvalues λ_n and eigenvectors $\vec{x}_n$:

$$\begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ c & 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$