PHOT 301: Quantum Photonics

Homework: Finite dimensional spaces and Dirac bra-ket notation

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Inner/outer products, bra's and kets

When we have a finite basis, i.e. our wavefunction lives in a finite n-dimensional Hilbert space, then this space is bijective isometric with $\mathbb{R}^n$ and we can always represent the wavefunction as a vector with n components. For example:

$$|b\rangle = \sum_{i=1}^{n} b_i |i\rangle \equiv \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \text{with} \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \ldots$$

In matrix formalism a ket $|b\rangle$ is represented by a column vector and a bra $\langle a|$ is a row vector. We can transform a ket into a bra by taking the Hermitian adjoint, this means transposing the vector and taking the complex conjugate of the elements:

$$|b\rangle = \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \langle a| = \vec{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}, \qquad \langle b| = |b\rangle^\dagger = \begin{pmatrix} b_1^* & b_2^* & b_3^* \end{pmatrix},$$

The inner product $\langle a|b\rangle$ becomes a matrix product of a row vector with a column vector, while an outer product $|b\rangle\langle a|$ is a product of a column with a row vector (resulting in a matrix). Assume in the following that kets $|a\rangle = (a_1, a_2, a_3)^\perp$ and $|b\rangle = (b_1, b_2, b_3)^\perp$ exist:

$$\langle a|b\rangle = \vec{a}^\dagger \cdot \vec{b} = \begin{pmatrix} a_1^* & a_2^* & a_3^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad |b\rangle\langle a| = \vec{b}\ \vec{a}^\dagger = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{pmatrix} a_1^* & a_2^* & a_3^* \end{pmatrix},$$

Exercises on inner(outer) products, perform the below exercises with following definitions:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |a\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |b\rangle = \begin{pmatrix} i \\ 1+2i \end{pmatrix}, \quad |c\rangle = \begin{pmatrix} 1-i \\ i \end{pmatrix}, \quad |d\rangle = \begin{pmatrix} -i \\ 0 \end{pmatrix}$$

$$\langle a|c\rangle = \qquad \langle d|1\rangle = \qquad \langle 1|c\rangle =$$

$$|a\rangle\langle a|c\rangle = \qquad |1\rangle\langle d| = \qquad \langle c|1\rangle\langle 1|c\rangle =$$

$$|d\rangle\langle b|2\rangle = \qquad |2\rangle\langle 1| = \qquad \langle a|a\rangle|d\rangle\langle c| =$$

$$(|a\rangle + |b\rangle)\langle c| = \qquad \langle 2|1\rangle\langle b| - \langle d| = \qquad |1\rangle^\dagger\langle b|b\rangle^* =$$

Projection operator

Using bra-ket notation the projection operator can be simple defined, for normalized $|\alpha\rangle$ as an outer product:

$$\hat{P}_\alpha = |\alpha\rangle\langle\alpha|$$

The projection operator projects any other vector $|b\rangle$ onto the direction of $|\alpha\rangle$:

$$\hat{P}_\alpha|b\rangle = (\langle\alpha|b\rangle)\,|\alpha\rangle$$

Exercises on the projection operator, perform the below exercises with following definitions:

$$|\alpha\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\ i\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad |2\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}, \quad |a\rangle = \begin{pmatrix}1\\ 1\end{pmatrix}, \quad |b\rangle = \begin{pmatrix}i\\ 1+2i\end{pmatrix}, \quad |c\rangle = \begin{pmatrix}1-i\\ i\end{pmatrix}.$$

$$\hat{P}_\alpha|1\rangle = \qquad \hat{P}_\alpha|2\rangle =$$

$$\hat{P}_\alpha|a\rangle = \qquad \hat{P}_1|c\rangle =$$

$$\hat{P}_2\hat{P}_1|1\rangle = \qquad \hat{P}_1\hat{P}_\alpha|c\rangle =$$

$$\hat{P}_1\hat{P}_2(|a\rangle + |b\rangle - |c\rangle) = \qquad \hat{P}_1(|a\rangle + |b\rangle - |c\rangle) =$$

$$\hat{P}_1|c\rangle + \hat{P}_2|c\rangle = \qquad \langle b|\hat{P}_1|b\rangle =$$

$$\langle 1|\hat{P}_1|1\rangle = \qquad \langle 1|\hat{P}_\alpha|1\rangle =$$