PHOT 301: Quantum Photonics

Project topics: project 1

Michaël Barbier, Fall semester (2024-2025)

Introduction

There are four projects to be performed during the PHOT 301 course of this semester. This file contains the project topics for the first project. The projects are meant to focus more on applied "real world" problems.

You can and are encouraged to work together on projects, further, you can ask help from me and Yağız (asking help will not influence your project grade). However, your project report and any figures containing plots and/or schematics should be made individually and not copied from others or online resources. Please cite any sources that you used and where you used them (you don't have to cite this document).

Type of report for project 1

The report should be between one and two pages (one sheet) including figures. Please ask help to your instructors on time, we might have not enough time to help you at the last day before the deadline of the report.

Grading of the project

This project will count for 10% of your grade. During this semester four projects will be made in total, corresponding to 40% of your total grade.

Project topics

Next is a list of problems out of which you can choose for your project together with their task description. You only have to solve one problem for your project. Please inform me if the problem description contains any errors or anything is unclear.

Problem 1: Angle-dependent transmission and reflection at a 1D step potential

Calculate the transmission and reflection coefficients of a propagating wave in a 2D plane which hits a potential step at $x = 0$:

$$V(x,y) = V(x) = \begin{cases} 0 x < 0 \\ V_0 x > 0 \end{cases}$$

Consider for this the time-independent Schrodinger equation and show that k_y is a constant of motion by separation of the variables. Therefore, assume that

$$\psi(x,y) = \phi(x)e^{ik_y y}$$

and fill this solution in into the time-independent Schrodinger equation:

$$\begin{aligned} \frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2} &= -\frac{2m}{\hbar^2}(E - V(x))\psi(x,y) \\ \Rightarrow e^{ik_y y}\frac{\partial^2\phi(x)}{\partial x^2} - k_y^2\phi(x)\frac{\partial^2 e^{ik_y y}}{\partial y^2} &= -\frac{2m}{\hbar^2}(E - V(x))\phi(x)e^{ik_y y} \\ \Rightarrow \frac{1}{\phi(x)}\frac{\partial^2\phi(x)}{\partial x^2} &= k_y^2 - \frac{2m}{\hbar^2}(E - V(x)) \\ \Rightarrow \frac{\partial^2\phi(x)}{\partial x^2} &= -\left[\frac{2m}{\hbar^2}(E - V(x)) - k_y^2\right]\phi(x) \end{aligned}$$

Then solve the Schrodinger equation for the solution of $\phi(x)$ and assume the wave function before and after the step at $x = 0$:

$$\begin{aligned} x < 0 \qquad & \phi(x) = Ae^{ikx} + Be^{-ikx} \\ x > 0 \qquad & \phi(x) = Ce^{iqx} \end{aligned}$$

with $k^2 = \frac{2m}{\hbar^2}E - k_y^2$ and $q^2 = \frac{2m}{\hbar^2}(E - V_0) - k_y^2$.

Be careful that in this case the transmission coefficient cannot be calculated from the amplitude of the transmitted wave in a trivial manner (because the wave vector $k \equiv k_x$ before the step is not equal to the wave vector $q \equiv q_x$ after the step). Use the fact that the transmission coefficient can be expressed in the reflection coefficient $T = 1 - R$ and that $R = |B|^2/|A|^2$

- Plot the transmission as function of the energy E and the angle $\theta = \arctan(k_y/k_x)$ with the horizontal of the incoming wave.
- Describe what happens when the angle θ becomes large such that $\frac{2m}{\hbar^2}(E - V_0) - k_y^2 < 0$.

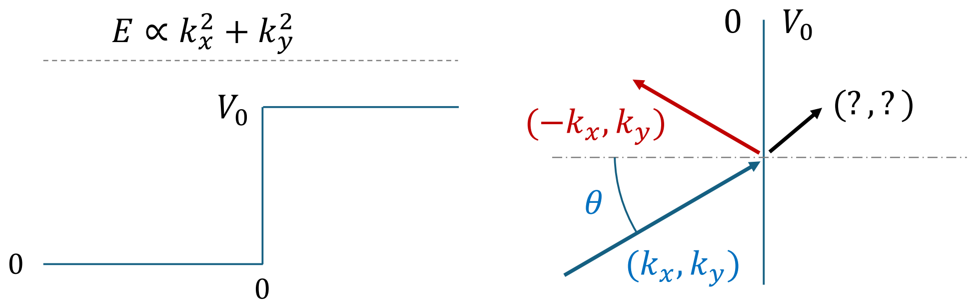

Figure 1: Transmission and reflection of a 2D propagating wave, coming from the left, incident on a 1D potential step given by $V(x > 0) = V_0$, under an angle $\theta = \arctan(k_y/k_x)$. **Left:** side view of the potential step. **Right:** top view of the potential step.

Problem 2: Transmission through a barrier

The transfer matrix method is an especially convenient method when having a piece-wise constant potential energy function $V(x)$. Assume the 1D staircase increasing potential barrier in the figure below, notice that the height increases as multiples of V and the width of each step of the staircase is a.

For the transfer matrix you assume propagating waves in every region of constant potential:

$$\psi_j(x) = A_j e^{ik_jx} + B_j e^{-ik_jx}, \text{ with } k_j = \sqrt{2m(E - V_j)}/\hbar$$

The continuity conditions of the wave function ψ and its derivative $\frac{d\psi}{dx}$ at the potential steps at a location x_j result in the transfer matrix between the coeffients:

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix} = M_j \begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} \quad \text{with} \quad M_j = \frac{1}{2k_j} \begin{pmatrix} -(k_{j+1} + k_j)e^{i(k_{j+1}-k_j)x_j} & (k_{j+1} - k_j)e^{-i(k_{j+1}+k_j)x_j} \\ (k_{j+1} - k_j)e^{i(k_{j+1}+k_j)x_j} & -(k_{j+1} + k_j)e^{-i(k_{j+1}-k_j)x_j} \end{pmatrix}$$

The transfer matrix can you help to link the coefficients for every region, use it to calculate the transmission coefficient T. Hereby remember that you can use $T = |A_4|^2/|A_0|^2$, where A_4 is the transmitted wave amplitude (because $k_0 = k_4$) and $B_4 = 0$.

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = M_0 M_1 M_2 M_3 \begin{pmatrix} A_4 \\ B_4 = 0 \end{pmatrix}$$

Perform the following tasks:

- Calculate the transmission coefficient T as function of the energy E of incoming propagating waves from the left (in 1D). This can be numerically.
- Plot the transmission T as function of the energy E.
- Plot and compare with the transmission through a single barrier of heigt $2V$ and width $3a$ (the "averaged" potential barrier).

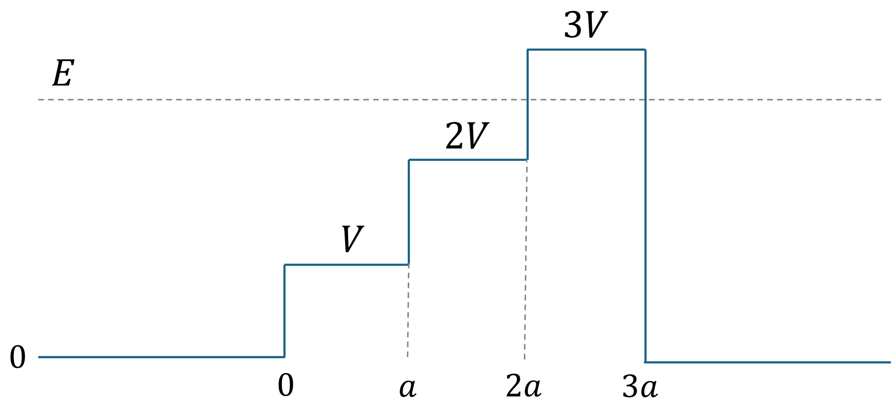

Figure 2: Potential barrier existing of three parts, of equal width a and increasing in height.

Problem 3: Visualizing a single electron in a quantum dot

Model a square 2D quantum dot with side length L and a potential energy function $V(x,y)$:

$$V(x,y) = \begin{cases} 0 & 0 < x < L \quad \& \quad 0 < y < L \\ +\infty & \text{otherwise} \end{cases}$$

The 2D quantum dot is here modeled as a 2D infinite well, the solutions are combinations of the 1D infinite well in x and y direction. Describe the time-independent solutions by factoring the wave function (separation of the variables).

$$\psi(x,y) = \psi_x(x)\psi_y(y)$$

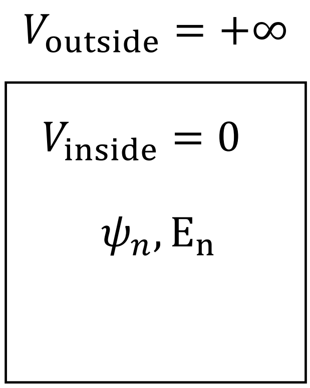

Figure 3: Square 2D quantum dot with side length L. Inside the quantum dot $V(x,y) = 0$, outside the potential is infinite.

Perform the following tasks:

- Derive the solutions for the eigenstates and eigenenergies of the time-independent problem.

- Plot the first 5 energy-levels of the quantum dot (in eV) as function of the side-length L of the quantum dot (in nm). Take the side-lengths of the quantum dot in the range $L = [10, 1000]$ nm.
- Visualize a superposition of eigenstates $\psi_n(x, y)$ as a contour or density plot. Take as an example $\psi(x, y) \propto \psi_{x,1}(x)\psi_{y,1}(y) + \psi_{x,2}(x)\psi_{y,1}(y)$, this is, a superposition of the ground state with the 2nd asymmetric eigenstate in x. Visualize both the wave function (time-independent solution) and the probability density function $|\psi|^2$.

Problem 4: Connected wells

Consider two quantum wells brought into close proximity such that a particle can tunnel between them. To simplify the model use an infinite well with a delta-function barrier in the middle of the well. The total potential energy function $V(x)$ is then given by the delta-function barrier with strength α inside the well $V(x) = \alpha\delta(x)$, while outside the well $V(x) = +\infty$. Use the analytical result for the stationary solutions from the attached notes: "calculation_connected_wells.pdf":

$$\psi_e(x) = A\sin(k|x|) + B\cos(kx) \qquad \text{with} \quad \frac{B}{A} = \tan(ka) = -(ka)\,\frac{\hbar^2}{m\alpha\, a}$$

$$\psi_o(x) = A\sin(kx) \qquad \text{with} \quad k = \frac{\sqrt{2mE}}{\hbar^2} \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Here the eigenstates $\psi_e(x)$ and $\psi_o(x)$ are the even and odd solutions. Remark that each of them represents multiple solutions, and that the eigenenergies of the even solutions are given by a transcendental equation.

Perform the following tasks:

- Numerically find the energy values (i.e., the eigenenergies) of the first three even eigenstates.
- How do the eigenenergy values of the even eigenstates change when increasing the potential barrier strength α: do they increase or decrease?
- What about the impact of α on the odd eigenenergies?

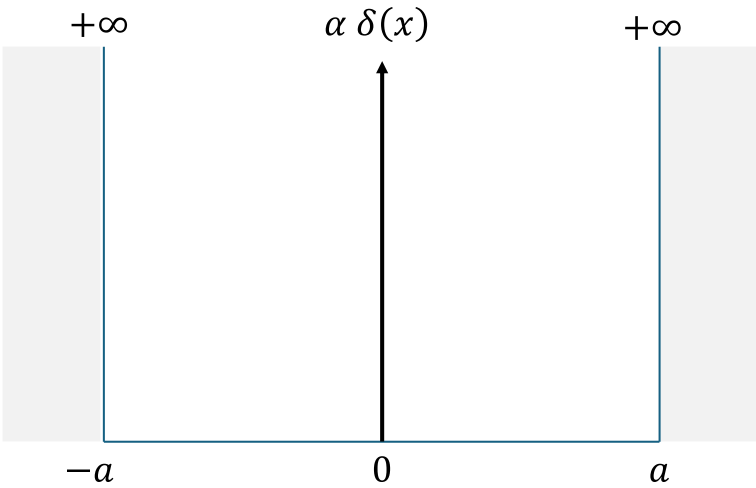

Figure 4: An infinite square well with a delta-function potential barrier in the middle. This can be seen as two connected quantum wells with the strength α of the delta-function potential barrier allowing tunneling between the wells.