PHOT 301: Quantum Photonics

Quiz 4: questions & solutions

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Exam questions

Grading: Each quiz counts for 15% of your total grade. Each question is valued equally in the score calculation.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

Hints given at the quiz:

For Hermitian $\hat{Q}$: $\hat{Q}\,\psi_n = q_n\,\psi_n \quad \Rightarrow \quad \exists c_n : \psi = \sum_n c_n \psi_n, \quad \langle Q \rangle = |c_n|^2 q_n$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y, \quad \hat{L}_+\,Y_l^m = \hbar\sqrt{(l-m)(l+m+1)}\,Y_l^{m+1}, \quad \hat{L}_-\,Y_l^m = \hbar\sqrt{(l+m)(l-m+1)}\,Y_l^{m-1}$$

Spherical harmonics: $\quad Y_0^0 = \dfrac{1}{2\sqrt{\pi}}, \quad Y_1^0 = \sqrt{\dfrac{3}{4\pi}}\cos\theta, \quad Y_1^{\pm 1} = \mp\sqrt{\dfrac{3}{8\pi}}\sin\theta\,e^{\pm i\phi},$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1), \quad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{\pm i\phi}, \quad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{\pm i2\phi}$$

Question 1: General uncertainty relation

The general uncertainty relation with operators $\hat{L}_x$ and $\hat{L}_y$ in a hydrogen atom:

$$\sigma^2_{L_x}\sigma^2_{L_y} \geq \left(\frac{1}{2i}\left\langle\left[\hat{L}_x, \hat{L}_y\right]\right\rangle\right)^2, \text{ where } \left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z \quad \text{and} \quad \hat{L}_z\psi_{nlm} = m\hbar\psi_{nlm}$$

(a) Calculate the value on the right-hand-side for the eigenstate $\psi_{nlm} = \psi_{211}$.
(b) Then assume the system is in superposition state $\psi = \alpha\psi_{2,1,1} + \beta\psi_{2,1,-1}$ with $|\alpha|^2 + |\beta|^2 = 1$. Express the expectation value $\langle L_z \rangle$ as function of α and β.

Solution (Q1)

(a) First we simplify the right-hand-side of the inequality:

$$\left(\frac{1}{2i}\left\langle\left[\hat{L}_x, \hat{L}_y\right]\right\rangle\right)^2 = \left(\frac{i\hbar}{2i}\left\langle\hat{L}_z\right\rangle\right)^2 = \frac{\hbar^2}{4}\langle\hat{L}_z\rangle^2$$

Then we use $\langle\hat{L}_z\rangle = \langle\psi_{211}|\hat{L}_z|\psi_{211}\rangle = \hbar\langle\psi_{211}|\psi_{211}\rangle = \hbar$, and we obtain:

$$\sigma_{L_x}^2\sigma_{L_y}^2 \geq \frac{\hbar^2}{4}\langle\hat{L}_z\rangle^2 = \frac{\hbar^4}{4}$$

(b) Since $\hat{L}_z$ is a Hermitian operator and has same eigenstates with the hydrogen atom we can expand its expectation value in the eigenvalues:

$$\langle L_z\rangle = |\alpha|^2\hbar + |\beta|^2(-\hbar) = \hbar(|\alpha|^2 - |\beta|^2)$$

Question 2: Probability & expectation

(a) Consider a hydrogen atom in the groundstate:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}\,e^{-r/a}, \qquad \text{with } a \text{ the Bohr radius.}$$

What is the probability for the electron to be at a distance r farther than $\frac{3}{2}a$?
(b) What is the expectation value for the energy $\langle H\rangle$ if the system is in superposition state: $\psi = \frac{1}{\sqrt{2}}(\psi_{100} + \psi_{210})$? *Hint:* Eigenenergies $E_n = -\text{Ry}/n^2 = -13.6/n^2$ eV.

Solution (Q2)

(a) The probability $P(r > 3a/2)$ is given by integrating the $|\psi_{100}|^2$ over the volume outside that radius:

$$\begin{aligned}
P(r > 3a/2) &= \int_{3a/2}^{+\infty} r^2|\psi_{100}|^2\,dr \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\phi \\
&= \int_{3a/2}^{+\infty} r^2\left|\frac{1}{\sqrt{\pi a^3}}e^{-r/a}\right|^2 dr\cdot 2\cdot 2\pi \\
&= \frac{4\pi}{\pi a^3}\int_{3a/2}^{+\infty} r^2 e^{-2r/a}\,dr \\
&= \frac{4}{a^3}\left[\left(-\frac{ar^2}{2} - \frac{a^2 2r}{4} - \frac{2a^3}{8}\right)e^{-2r/a}\right]\Bigg|_{3a/2}^{+\infty} \\
&= \frac{4}{a^3}\left[0 - \left(-\frac{9a^3}{8} - \frac{6a^3}{8} - \frac{2a^3}{8}\right)e^{-3}\right] = \frac{17}{2e^3}
\end{aligned}$$

(b) The expectation value for the energy can be expanded in its eigenvalues:

$$\langle H \rangle = \sum_n |c_n|^2 E_n = \frac{1}{2}E_1 + \frac{1}{2}E_2 = -\frac{1}{2}\left(13.6 + \frac{13.6}{2^2}\right)\,\text{eV} = -\frac{17}{2}\,\text{eV} = -\frac{5}{8}\text{Ry}$$

Question 3: Angular momentum

Consider the angular part $Y_l^m(\theta,\phi)$ (spherical harmonics) of the hydrogen atom solutions $\psi_{nlm} = R_{nl}(r)Y_l^m(\theta,\phi)$.

(a) Calculate $(\hat{L}_+^2 + \hat{L}_-^2)Y_2^0(\theta,\phi)$, and simplify the resulting expression as much as possible.
(b) Consider then the state given by wavefunction $\Psi(\theta,\phi,t=0) = \frac{1}{5}(3Y_2^1 + 4Y_1^1)$. Derive an expression for the probability density function $|\Psi(\theta,\phi,t)|^2$. Does it depend on time?

Solution (Q3)

(a) The resulting function is:

$$\begin{aligned}\left(\hat{L}_+^2 + \hat{L}_-^2\right) Y_2^0 &= \hat{L}_+^2 Y_2^0 + \hat{L}_-^2 Y_2^0 \\ &= \hbar\sqrt{(2)(3)}\hat{L}_+ Y_2^1 + \hbar\sqrt{(2)(3)}\hat{L}_- Y_2^{-1} \\ &= \hbar^2\sqrt{6}\sqrt{(1)(4)}Y_2^2 + \hbar^2\sqrt{6}\sqrt{(1)(4)}Y_2^{-2} \\ &= 2\sqrt{6}\hbar^2\left(Y_2^2 + Y_2^{-2}\right)\end{aligned}$$

Then we fill in the expressions for the spherical harmonics:

$$\begin{aligned}2\sqrt{6}\hbar^2\left(Y_2^2 + Y_2^{-2}\right) &= 2\sqrt{6}\hbar^2\sqrt{\frac{15}{32\pi}}\sin^2\theta\,\left(e^{i2\phi} + e^{-i2\phi}\right) \\ &= \hbar^2 3\sqrt{\frac{5}{\pi}}\sin^2\theta\,\cos(2\phi)\end{aligned}$$

In the original question there was a missing $\hbar$ in the given formulas, answers with or without are accepted.

(b) We add time factors to the eigenstates: $Y_2^1 e^{-i\omega_m}$ and $Y_1^1 e^{-i\omega_n}$ with $\omega_{n,m} = E_{n,m}/\hbar$, and assume further $\Delta\omega = \omega_m - \omega_n$.

The probability density function is given by

$$
\begin{aligned}
|\Psi(\theta,\phi,t)|^2 &= \left|\frac{1}{5}(3Y_2^1 e^{-i\omega_m t} + 4Y_1^1 e^{-i\omega_n t})\right|^2 \\
&= \frac{9}{25}\left|Y_2^1\right|^2 + \frac{16}{25}\left|Y_1^1\right|^2 + \frac{12}{25}\left({Y_1^1}^* Y_2^1 e^{-i\Delta\omega t} + {Y_2^1}^* Y_1^1 e^{i\Delta\omega t}\right) \\
&= \frac{9}{25}\left|Y_2^1\right|^2 + \frac{16}{25}\left|Y_1^1\right|^2 + \frac{24}{25}\cos(\Delta\omega t)\left(\frac{3\sqrt{5}}{8\pi}\sin^2\theta\cos\theta\right) \\
&= \frac{9}{25}\frac{15}{8\pi}\sin^2\theta\cos^2\theta + \frac{16}{25}\frac{3}{8\pi}\sin^2\theta + \frac{24}{25}\cos(\Delta\omega t)\left(\frac{3\sqrt{5}}{8\pi}\sin^2\theta\cos\theta\right) \\
&= \frac{3}{25\cdot 8\pi}\sin^2\theta\left(45\cos^2\theta + 16 + 24\sqrt{5}\cos\theta\cos(\Delta\omega t)\right)
\end{aligned}
$$

which is independent of time if $\Delta\omega = 0$.

For the hydrogen atom the energy-levels depend on the principle quantum number n, belonging to the radial part $R_{nl}(r)$, which we implicitly assumed to be same for both when making the superposition. Therefore, it is reasonable to put $\Delta\omega = 0$.