PHOT 301 Fall (2025-2026) **Quiz 4** Date: 18 December 2025
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Student Name:

Grading: Each quiz counts for 15% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam. Please tell if any question is unclear or ambiguous.

Question 1: General uncertainty relation

The general uncertainty relation with operators $\hat{L}_x$ and $\hat{L}_y$ in a hydrogen atom:

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\left\langle\left[\hat{L}_x, \hat{L}_y\right]\right\rangle\right)^2, \text{ where } \left[\hat{L}_x, \hat{L}_y\right] = i\hbar \hat{L}_z \quad \text{and} \quad \hat{L}_z \psi_{nlm} = m\hbar\psi_{nlm}$$

(a) Calculate the value on the right-hand-side for the eigenstate $\psi_{nlm} = \psi_{211}$.
(b) Then assume the system is in superposition state $\psi = \alpha\psi_{2,1,1} + \beta\psi_{2,1,-1}$ with $|\alpha|^2 + |\beta|^2 = 1$. Express the expectation value $\langle L_z \rangle$ as function of α and β.

Question 2: Probability & expectation

(a) Consider a hydrogen atom in the groundstate:

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \qquad \text{with } a \text{ the Bohr radius.}$$

What is the probability for the electron to be at a distance r farther than $\frac{3}{2}a$?
(b) What is the expectation value for the energy $\langle H \rangle$ if the system is in superposition state: $\psi = \frac{1}{\sqrt{2}}(\psi_{100} + \psi_{210})$? *Hint:* Eigenenergies $E_n = -\text{Ry}/n^2 = -13.6/n^2$ eV.

Question 3: Angular momentum

Consider the angular part $Y_l^m(\theta, \phi)$ (spherical harmonics) of the hydrogen atom solutions $\psi_{nlm} = R_{nl}(r) Y_l^m(\theta, \phi)$.

(a) Calculate $(\hat{L}_+^2 + \hat{L}_-^2) Y_2^0(\theta, \phi)$, and simplify the resulting expression as much as possible.
(b) Consider then the state given by wavefunction $\Psi(\theta, \phi, t = 0) = \frac{1}{5}(3Y_2^1 + 4Y_1^1)$. Derive an expression for the probability density function $|\Psi(\theta, \phi, t)|^2$. Does it depend on time?

Hints: For Hermitian $\hat{Q}$: $\hat{Q}\,\psi_n = q_n\,\psi_n \quad \Rightarrow \quad \exists c_n : \psi = \sum_n c_n \psi_n, \quad \langle Q \rangle = |c_n|^2 q_n$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y, \quad \hat{L}_+ Y_l^m = \hbar\sqrt{(l-m)(l+m+1)}\, Y_l^{m+1}, \quad \hat{L}_- Y_l^m = \hbar\sqrt{(l+m)(l-m+1)}\, Y_l^{m-1}$$

Spherical harmonics: $Y_0^0 = \frac{1}{2\sqrt{\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi},$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1), \quad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta\, e^{\pm i\phi}, \quad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{\pm i2\phi}$$