PHOT 301: Quantum Photonics

Quiz 3: questions & solutions

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Exam questions

Grading: Each quiz counts for 15% of your total grade. Each question is valued equally in the score calculation.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

Question 1: Delta-function Potential

Consider a system with delta-function potential barrier $V(x) = \alpha\delta(x)$ with $\alpha > 0 \in \mathbb{R}$.

(a) Derive an expression for the reflection coefficient R (not only the end result, please write down the derivation).
(b) If you increase the barrier strength fourfold: $\alpha \longrightarrow 4\alpha$, by what factor β should you increase $E \longrightarrow \beta E$ to retain the same reflection coefficient? Explain.

Solution (Q1)

(a) To derive the reflection coefficient R we assume following solutions left (region I) and right (region II) from the δ-function potential:

$$\psi_I(x) = A\,e^{ikx} + B\,e^{-ikx}$$
$$\psi_{II}(x) = F\,e^{ikx}$$

Where $k = \sqrt{2mE}/\hbar$ and we assume no incident wave from the right. The reflection coefficient is given by: $R = \frac{|B|^2}{|A|^2}$. To obtain ratio B/A we apply first continuity of the wave function $\psi_I(0) = \psi_{II}(0)$:

$$\psi_I(0) = \psi_{II}(0) \quad \Rightarrow \quad A + B = F$$

The derivative of the wave function is not continuous at zero (because of the δ-function):

$$\psi_I'(x) = \left(A\,e^{ikx} + B\,e^{-ikx}\right)' \quad \Rightarrow \quad \psi_I'(0) = ik(A-B)$$
$$\psi_{II}'(x) = \left(F\,e^{ikx}\right)' \quad \Rightarrow \quad \psi_{II}'(0) = ikF$$

Therefore the difference in derivative at zero is $\Delta\psi' = \psi_{II}'(0) - \psi_I'(0) = ik(F - A + B)$.

Now we integrate the Schrodinger equation over a small interval around zero: $[-\epsilon, \epsilon]$:

$$\begin{aligned}
&\frac{-\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{\partial^2\psi(x)}{\partial x^2}dx + \alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx = E\int_{-\epsilon}^{\epsilon}\psi(x)\,dx \\
\Rightarrow\ &\frac{-\hbar^2}{2m}\left[\frac{\partial\psi(x)}{\partial x}\right]\Big|_{-\epsilon}^{\epsilon} + \alpha\psi(0) = 0 \\
\Rightarrow\ &\left[\frac{\partial\psi(x)}{\partial x}\right]\Big|_{-\epsilon}^{\epsilon} = \frac{2m\alpha}{\hbar^2}\psi(0)
\end{aligned}$$

Since $\psi(0) = A + B = F$, and for infinitesimally small ϵ: $\Delta\psi' = \left[\frac{\partial\psi(x)}{\partial x}\right]\Big|_{-\epsilon}^{+\epsilon}$ we obtain:

$$\Delta\psi' = \frac{2m\alpha}{\hbar^2}(A+B) = ik(F - A + B) = ik(A + B - A + B) = i2kB$$

Rearranging to obtain factor $\beta \equiv B/A$:

$$\begin{aligned}
&\frac{2m\alpha}{\hbar^2}(A+B) = i2kB \\
\Rightarrow\ &\frac{2m\alpha}{\hbar^2}(1+\beta) = i2k\beta \\
\Rightarrow\ &\frac{2m\alpha}{\hbar^2}\left(\frac{1}{\beta}+1\right) = i2k \\
\Rightarrow\ &\left(\frac{1}{\beta}+1\right) = \frac{i2k\hbar^2}{2m\alpha} \\
\Rightarrow\ &\beta = \frac{1}{\frac{i2k\hbar^2}{2m\alpha} - 1} \\
\Rightarrow\ &\beta = \frac{1}{i\gamma - 1}
\end{aligned}$$

Where we further defined $\gamma \equiv \frac{2k\hbar^2}{2m\alpha}$. The reflection coefficient is then given by:

$$R = \frac{|B|^2}{|A|^2} = |\beta|^2 = \frac{1}{|i\gamma - 1|^2} = \frac{1}{1+\gamma^2} = \frac{1}{1 + \frac{k^2\hbar^4}{m^2\alpha^2}} = \frac{1}{1 + \frac{2E\hbar^2}{m\alpha^2}}$$

Where we filled in $k = \sqrt{2mE}/\hbar$.

(b) From above resulting formula we can see that if we multiply $\alpha \longrightarrow 4\alpha$ the second term in the denominator becomes:

$$\frac{2E\hbar^2}{m\alpha^2} \longrightarrow \frac{1}{16}\frac{2E\hbar^2}{m\alpha^2}$$

To compensate we should increase the energy $E \longrightarrow 16E$.

Question 2: Operator and Commutators

(a) Calculate the commutator $[e^{ix^2}, \frac{1}{x}\frac{\partial}{\partial x}]$.
(b) Functions $f(x) = e^{-2x}$ and $g(x) = \sin(3x)$ are eigenfunctions of the operator $\hat{Q} = \frac{d^2}{dx^2}$. What are the corresponding eigenvalues?

Solution (Q2)

(a) The commutator $[e^{ix^2}, \frac{1}{x}\frac{\partial}{\partial x}]$ can be calculated using a test-function $f(x)$:

$$\begin{aligned}
\left[e^{ix^2}, \frac{1}{x}\frac{\partial}{\partial x}\right] f &= e^{ix^2}\frac{1}{x}\frac{\partial f}{\partial x} - \frac{1}{x}\frac{\partial(e^{ix^2} f)}{\partial x} \\
&= e^{ix^2}\frac{1}{x}\frac{\partial f}{\partial x} - \frac{1}{x}\left[i2xe^{ix^2} f + e^{ix^2}\frac{\partial f}{\partial x}\right] \\
&= e^{ix^2}\frac{1}{x}\frac{\partial f}{\partial x} - i2e^{ix^2} f - \frac{1}{x}e^{ix^2}\frac{\partial f}{\partial x} \\
&= -i2e^{ix^2} f
\end{aligned}$$

Removing the test-function we get the resulting commutator:

$$\left[e^{ix^2}, \frac{1}{x}\frac{\partial}{\partial x}\right] = -i2e^{ix^2}$$

(b) We obtain the eigenvalues for $f(x) = e^{-2x}$ and $g(x) = \sin(3x)$ by simple applying the operator:

$$\begin{aligned}
\hat{Q}f(x) &= \frac{d^2 e^{-2x}}{dx^2} = -2\,\frac{de^{-2x}}{dx} = 4e^{-2x} = 4f(x) \\
\hat{Q}g(x) &= \frac{d^2 \sin(3x)}{dx^2} = 3\,\frac{d\cos(3x)}{dx} = -9\sin(3x) = -9\,g(x)
\end{aligned}$$

Therefore the eigenvalues for $f(x)$ and $g(x)$ are 4 and -9, respectively.

Question 3: Matrix Representation

Consider a three-state system where an operator $\hat{Q}$ and a state $|\alpha\rangle$ are represented in matrix notation by:

$$Q = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad |\alpha\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

(a) What are the eigenvalues of Q^2?
(b) Calculate the expectation value $\langle\alpha|\hat{Q}|\alpha\rangle$.

Solution (Q3)

(a) The matrix Q^2 is the unity matrix:

$$Q^2 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the eigenvalue equation becomes trivial: all three eigenvalues are equal to one: $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The eigenvectors (not required for this question) can be chosen as three linear independent vectors, for example:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) The expectation value $\langle\alpha|\hat{Q}|\alpha\rangle$ is given by:

$$\begin{aligned} \langle\alpha|\hat{Q}|\alpha\rangle &= \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} i \\ 0 \\ -2 \end{pmatrix} \\ &= \frac{1}{5}(0 + 0 - 4) = -\frac{4}{5}, \end{aligned}$$

where we used the fact that the bra is the Hermitian adjoint of the given ket $|\alpha\rangle$:

$$\langle\alpha| = |\alpha\rangle^\dagger = \frac{1}{5} \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$$