PHOT 301: Quantum Photonics

Quiz 2: questions & solutions

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Exam questions

Grading: Each quiz counts for 15% of your total grade. Each question is valued equally in the score calculation.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

This document contains both the problems and their solutions. Following hints were given at the end of the exam form:

Question 1: Harmonic oscillator

Consider the harmonic oscillator in the first excited state (already normalized) defined with $x \in \mathbb{R}$:

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2} = \sqrt{2}\,\pi^{-1/4}\,\beta^{3/2} x\, e^{-\beta^2 x^2/2} \qquad \text{with } \beta = \sqrt{\frac{m\omega}{\hbar}}$$

(a) Calculate the expectation value for the position $\langle x \rangle$.
(b) Then calculate the expectation value $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

Hint: Expectation value for a function $f(x)$: $\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\, |\psi|^2\, dx$.

Solution (Q1)

(a) The expectation value for the position can be derived as:

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\, |\psi|^2\, dx = \int_{-\infty}^{+\infty} x 2\pi^{-1/2}\beta^3 x^2\, e^{-\beta^2 x^2}\, dx = 2\pi^{-1/2}\beta^3 \int_{-\infty}^{+\infty} x^3\, e^{-\beta^2 x^2}\, dx = 0$$

The integral is zero, since the integrad $x^3\, e^{-\beta^2 x^2}$ is an odd function in x. Therefore $\langle x \rangle = 0$.

(b) For the expectation value, or standard deviation $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ we know from part (a) that $\langle x \rangle = 0$. Therefore, we only still require $\langle x^2 \rangle$:

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{+\infty} x^2\, |\psi|^2\, dx \\
&= \int_{-\infty}^{+\infty} x^2 2\pi^{-1/2} \beta^3 x^2\, e^{-\beta^2 x^2}\, dx \\
&= 2\pi^{-1/2}\beta^3 \int_{-\infty}^{+\infty} x^4\, e^{-\beta^2 x^2}\, dx \\
&= 2\pi^{-1/2}\beta^3 \frac{3\sqrt{\pi}}{4(\beta^2)^{5/2}} = \frac{3}{2}\beta^{-2}
\end{aligned}$$

Where for the last integral we used the known solution:

$$\int_{-\infty}^{+\infty} x^4\, e^{-ax^2}\, dx = \frac{3\sqrt{\pi}}{4a^{5/2}}$$

The resulting value is:

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{3}{2}\beta^{-2}} = \sqrt{\frac{3}{2}\frac{\hbar}{m\omega}}$$

Since $\hbar$ has units $[\hbar] = \mathrm{J\ s} = \mathrm{kg\ m}^2/\mathrm{s}$, $[m] = \mathrm{kg}$, and $[\omega] = \mathrm{s}^{-1}$, the units of σ are $[\sigma] = \mathrm{m}$ as expected.

Question 2: Ladder operators

Assume a particle in a 1D harmonic oscillator has following wave function:

$$\psi(x) = A\left[(\hat{a}_+^2 + \hat{a}_+\hat{a}_-)\psi_0 + \hat{a}_+\psi_3\right]$$

(a) Apply the ladder operators so you get an expression in only the eigenstates of the harmonic oscillator: $\psi_n(x)$. No ladder operators should appear.
(b) Derive the normalization constant A.

Hint: Eigenstates are orthonormal, and $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$, $\quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$.

Solution (Q2)

(a) By applying the ladder operators we obtain:

$$\psi(x) = A\left[(\hat{a}_+^2 + \hat{a}_+\hat{a}_-)\psi_0 + \hat{a}_+\psi_3\right] \quad = A\left[\sqrt{2}\sqrt{1}\psi_2 + 0 + \sqrt{4}\psi_4\right] = A\left[\sqrt{2}\psi_2 + 2\psi_4\right]$$

(b) The normalization constant A we find by putting the total probability equal to one:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx \\
&= |A|^2 \int_{-\infty}^{+\infty} (\sqrt{2}\psi_2^* + 2\psi_4^*)(\sqrt{2}\psi_2 + 2\psi_4) \, dx \\
&= |A|^2 \int_{-\infty}^{+\infty} \left[2|\psi_2|^2 + 4|\psi_4|^2 + \sqrt{2}2\psi_2\psi_4^* + \sqrt{2}2\psi_2^*\psi_4\right] \, dx \\
&= |A|^2 \left[2\int_{-\infty}^{+\infty} |\psi_2|^2 \, dx + 4\int_{-\infty}^{+\infty} |\psi_4|^2 \, dx + 2\sqrt{2}\int_{-\infty}^{+\infty} \psi_2\psi_4^* \, dx + 2\sqrt{2}\int_{-\infty}^{+\infty} \psi_2^*\psi_4 \, dx\right] \\
&= |A|^2 \, [2 + 4 + 0 + 0] = 6|A|^2
\end{aligned}$$

Where we used the orthonormality of the eigenstates: $\int_{-\infty}^{+\infty} \psi_m^*\psi_n \, dx = \delta_{mn}$. If we choose A positive and real-valued we obtain $A = 1/\sqrt{6}$.

Question 3: Free particle: Gaussian wave packet

A free particle has following wave function at time zero:

$$\Psi(x, 0) = \psi(x) = Ae^{-ax^2}$$

(a) Derive the normalization constant A.
(b) Apply the Fourier transform to obtain $\phi(k)$. This is: $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x, 0)e^{-ikx} \, dx$.

Hint: You can make use of the known integral: $\int_{-\infty}^{+\infty} e^{-(ax^2+bx)} \, dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$ where "a" and "b" are constants which can be complex values.

Solution (Q3)

(a) The normalization constant A is found by putting the total probability equal to one:

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = |A|^2 \int_{-\infty}^{+\infty} e^{-2ax^2} \, dx = |A|^2 \frac{\sqrt{\pi}}{\sqrt{2a}}$$

Therefore, when choosing A real-valued and positive we obtain $A = \left(\frac{2a}{\pi}\right)^{1/4}$.

(b) The Fourier transform gives us $\phi(k)$:

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A e^{-ax^2} e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{+\infty} e^{-ax^2 - ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} e^{-k^2/4a} \\
&= \frac{1}{(2a\pi)^{1/4}} e^{-k^2/4a}
\end{aligned}$$

Where we used the identity:

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\, dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$$