PHOT 301: Quantum Photonics

Quiz 1: questions & solutions

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Exam questions

Grading: Each quiz counts for 15% of your total grade. Each question is valued equally in the score calculation.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the quiz is 1 hour.

This document contains both the problems and their solutions. Following hints were given at the end of the exam form:

Hints for questions 2 and 3: The eigenstates of the infinite well $\psi_n(x)$ have eigenenergies $E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$. The expansion $\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$ has coefficients:

$$c_n = \int_0^L \psi_n^* \, \psi \, dx, \qquad \text{where} \qquad \sum_{n=1}^{\infty} |c_n|^2 = 1 \quad \text{and} \quad \langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

Question 1: Wave functions

Consider the time-independent wave function $\psi(x)$ defined with $x \in \mathbb{R}$:

$$\psi(x) = A \, \frac{1}{\sqrt{x^2 + 9}}$$

(a) First calculate the normalization constant A of the wave function.
(b) Then calculate the probability to find the particle in interval $[-3, 3]$.

Solution (Q1)

(a) The total probability should be equal to one:

$$1 = |A|^2 \int_{-\infty}^{\infty} |\psi(x)|^2\, dx = |A|^2 \int_{-\infty}^{+\infty} \left|\frac{1}{\sqrt{x^2+9}}\right|^2 dx = |A|^2 \int_{-\infty}^{\infty} \frac{1}{x^2+9}\, dx$$
$$= |A|^2 \frac{1}{3}\left[\arctan\left(\frac{x}{3}\right)\right]\Big|_{-\infty}^{+\infty} = |A|^2 \frac{1}{3}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = |A|^2 \frac{\pi}{3}$$

And thus $A = \sqrt{\frac{3}{\pi}}$ when we choose it real and positive.

(b) The probability within the interval $[-3, 3]$ is calculated as $P(x \in [-3,3]) = \int_{-3}^{3} |\psi(x)|^2\, dx$ leading to:

$$P(x \in [-3,3]) = |A|^2 \int_{-3}^{3} |\psi(x)|^2\, dx = |A|^2 \int_{-3}^{3} dx = |A|^2 \int_{-3}^{3} \frac{1}{x^2+9}\, dx$$
$$= |A|^2 \frac{1}{3}\left[\arctan\left(\frac{x}{3}\right)\right]\Big|_{-3}^{3} = |A|^2 \frac{1}{3}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{3}{\pi}\frac{\pi}{6} = \frac{1}{2}$$

Question 2: Time evolution

Assume a particle in the groundstate of an infinite well of width L has following wave function at time zero:

$$\Psi(x,0) = \psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$$

(a) Write down both the expression for the time-dependent wave function $\Psi(x,t)$ and the probability density function $|\Psi(x,t)|^2$.
(b) The time-dependent wave function is a periodic function: $\Psi(x,t) = \Psi(x, t+T)$ with period T, derive an expression for T.

Solution (Q2)

(a) The time-dependent wave function $\Psi(x,t)$ is obtained by adding the time-dependency factor $e^{-iEt/\hbar}$, where for the groundstate $E = E_1 = \frac{\hbar^2\pi^2}{2mL^2}$, resulting in:

$$\Psi(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-iE_1 t/\hbar} = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-i\frac{\hbar\pi^2}{2mL^2}t}$$

The probability density function is given by:

$$|\Psi(x,t)|^2 = \left|\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-iE_1 t/\hbar}\right|^2 = \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right)$$

(b) The period T of the time-dependent wave function can be derived as (here we define $\omega_1 \equiv E_1/\hbar$):

$$\begin{aligned} \Psi(x,t) &= \Psi(x,t+T) \\ \Leftrightarrow \quad \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\, e^{-i\omega_1 t} &= \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\, e^{-i\omega_1 (t+T)} \\ \Leftrightarrow \quad 1 &= e^{-i\omega_1 T} \end{aligned}$$

Therefore $\omega_1 T$ should be a multiple of 2π and period $T = 2\pi/\omega_1 = 2\pi\hbar/E_1$.

Question 3: Infinite well

A particle in an infinite well with width L is in a superposition state at time $t = 0$:

$$\Psi(x,0) = \psi(x) = \frac{1}{\sqrt{5}}\left(\psi_2(x) - 2\psi_{10}(x)\right)$$

(a) What is the difference in energy $E_{10} - E_2$ between the eigenstates?
(b) What is the expectation value of the Hamiltonian $\langle \hat{H} \rangle$.

Solution (Q3)

(a) The eigenenergies of the infinite well $E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2} = n^2\, E_1$, and thus:

$$E_{10} - E_2 = 10^2\, E_1 - 2^2 E_1 = 96\, E_1$$

(b) The expectation value of the Hamiltonian $\langle \hat{H} \rangle$ can be calculated as follows:

$$\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

We know the coefficients c_n since the superposition is given, the only nonzero coefficients are: $c_2 = \frac{1}{\sqrt{5}}$ and $c_{10} = -\frac{2}{\sqrt{5}}$. Therefore:

$$\langle \hat{H} \rangle = |c_2|^2 E_2 + |c_{10}|^2 E_{10} = \frac{1}{5}\, 4E_1 + \frac{4}{5}\, 100 E_1 = \frac{404}{5} E_1$$