PHOT 301: Quantum Photonics

Midterm exam questions & solutions

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General information on the exam

Grading: This midterm exam will count for 10% of your total grade. Together with the projects which count for 40%, and the final exam that will account for 50%, your total grade for the course will be determined.

Exam type: The midterm exam consists of 7 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation: when you have to answer multiple subproblems each of the subtasks is given a score out of 5 points. For each question the sub-scores are then averaged.

Question 1: wave function in the half plane

Consider the following wave function defined on $x \in [0, +\infty]$:

$$\psi(x) = Axe^{-\frac{\pi x^2}{2}}$$

with A the normalization constant.

- *(1/3 of points)* First calculate the normalization constant A of the wave function.

- *(1/3 of points)* Then calculate the expectation value for the position operator $\langle \hat{x} \rangle \in [0, +\infty]$.
- *(1/3 of points)* Afterwards calculate the variance: $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 - 2\langle x \rangle x + \langle x \rangle^2 \rangle$.

Hint: For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of an operator $\hat{Q}$ is given by:

$$\langle \hat{Q} \rangle = \int_a^b \psi(x)^* \, \hat{Q} \, \psi(x) dx.$$

You also can make use of the following definite integrals:

$$\int_0^\infty e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{a}}, \qquad \int_0^\infty x e^{-ax^2} \, dx = \frac{1}{2a}, \qquad \int_0^\infty x^2 e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{4a^{3/2}},$$

$$\int_0^\infty x^3 e^{-ax^2} \, dx = \frac{1}{2a^2}, \qquad \int_0^\infty x^4 e^{-ax^2} \, dx = \frac{3\sqrt{\pi}}{8a^{5/2}}.$$

Solution (Q1)

- The normalization constant $A = 2\sqrt{\pi}$ as can be seen from:

$$1 = |A|^2 \int_0^\infty |x e^{-\pi x^2/2}|^2 \, dx = |A|^2 \int_0^\infty x^2 \, e^{-\pi x^2} \, dx = |A|^2 \frac{\sqrt{\pi}}{4\pi^{3/2}} = \frac{|A^2|}{4\pi}$$

- The expectation value for the position $\langle x \rangle$:

$$\langle x \rangle = |A|^2 \int_0^\infty x^3 e^{-\pi x^2} \, dx = 4\pi \, \frac{1}{2\pi^2} = \frac{2}{\pi}$$

- The variance $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$ is given by:

$$\begin{aligned}
\langle (x - \langle x \rangle)^2 \rangle &= \langle x^2 - 2\langle x \rangle \, x + x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \\
&= |A|^2 \int_0^\infty x^4 e^{-\pi x^2} \, dx - \langle x \rangle^2 \\
&= 4\pi \frac{3\sqrt{\pi}}{8\sqrt{\pi}^5} - \frac{4}{\pi^2} \\
&= \frac{1}{2\pi^2} \, (3\pi - 8)
\end{aligned}$$

Question 2: Harmonic oscillator

Assume a particle in a harmonic oscillator potential is initially in the state:

$$\Psi(x,0) = A\ (4\psi_0(x) + 3\psi_1(x))$$

with $\psi_n(x)$ the eigenstates of the harmonic oscillator.

- *(1/2 of points)* Find the normalization factor A of the wave function.
- *(1/2 of points)* Then show that the probability density function $|\Psi(x,t)|^2$ contains a time-dependent term which is proportional to $\cos(\omega t)$ by writing out its definition.

Hint: You can assume that the eigenstates are chosen to be real-valued and the eigenenergies corresponding to the the 1D harmonic oscillator (orthonormal) eigenstates: $\psi_n(x)$, are of the form:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Solution (Q2)

- The normalization factor $A = \frac{1}{5}$ is calculated as follows:

$$\begin{aligned}1 = |A|^2 \int_{-\infty}^{\infty} |\Psi(x,0)|^2\, dx &= |A|^2 \int_{-\infty}^{\infty} |4\psi_0(x) + 3\psi_1(x)|^2\, dx \\ &= |A|^2 \int_{-\infty}^{\infty} (16|\psi_0(x)|^2 + 9|\psi_1(x)|^2)\, dx = |A|^2\, 25\end{aligned}$$

- The probability density of $|\Psi(x,t)|^2$ can be written as:

$$\begin{aligned}\Psi(x,t) &= A\left[4\psi_0(x)\, e^{-iE_0t/\hbar} + 3\psi_1(x)\, e^{-iE_1t/\hbar}\right] = A\left[4\psi_0(x)\, e^{-i\omega t/2} + 3\psi_1(x)\, e^{-i3\omega t/2}\right] \\ \Rightarrow |\Psi(x,t)|^2 &= |A|^2 \int_{-\infty}^{\infty} |4\psi_0(x) + 3\psi_1(x)|^2\, dx \\ &= |A|^2 \left(4\psi_0(x)\, e^{i\omega t/2} + 3\psi_1(x)\, e^{i3\omega t/2}\right)\left(4\psi_0(x)\, e^{-i\omega t/2} + 3\psi_1(x)\, e^{-i3\omega t/2}\right) \\ &\quad \text{since } \psi_n(x) \text{ real:} \\ &= \frac{16}{25}|\psi_0(x)|^2 + \frac{9}{25}|\psi_1(x)|^2 + \frac{12}{25}\, 2\, \psi_0(x)\psi_1(x)\Re\left\{e^{-i\omega t}\right\} \\ &= \frac{16}{25}|\psi_0(x)|^2 + \frac{9}{25}|\psi_1(x)|^2 + \frac{24}{25}\, \psi_0(x)\psi_1(x)\cos(\omega t)\end{aligned}$$

where the last mixing term contains the time dependency and is oscillating with frequency $f = \omega/2\pi$.

Question 3: Expansion in eigenstates

For an infinite square well with width L the solutions can be written in the form:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}, \quad \text{with } n = 1, 2, 3, ...$$

- Derive the coefficients c_1 and c_2 for the wave function $\psi(x) = c_1\psi_1 + c_2\psi_2$, if you know that the expectation value for the energy $\langle \hat{H} \rangle = 3E_1$.

Hint: For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of an operator $\hat{Q}$ is given by:

$$\langle \hat{Q} \rangle = \int_a^b \psi(x)^* \, \hat{Q} \, \psi(x) dx.$$

Solution (Q3)

$$\begin{aligned}
\langle \hat{H} \rangle &= \langle \psi | \, \hat{H} \, | \psi(x) \rangle \\
&= (c_1^* \langle \psi_1 | + c_2^* \langle \psi_2 |) \, \hat{H} \, (c_1 | \psi_1 \rangle + c_2 | \psi_2 \rangle) \\
&= (c_1^* \langle \psi_1 | + c_2^* \langle \psi_2 |) \, (c_1 E_1 | \psi_1 \rangle + c_2 (4 \, E_1) | \psi_2 \rangle) \\
&= |c_1|^2 E_1 + |c_2|^2 4E_1 \\
&= |c_1|^2 E_1 + (1 - |c_1|^2) 4E_1 \\
\implies 3E_1 &= |c_1|^2 E_1 + (1 - |c_1|^2) 4E_1 \\
\implies |c_1|^2 &= 1/3, \qquad |c_2|^2 = 1 - |c_1|^2 = 2/3 \\
\implies c_1 &= 1/\sqrt{3}, \qquad c_2 = \sqrt{2/3}
\end{aligned}$$

Question 4: Transmission through a step potential

Consider propagating waves in a 1D system coming from the left, incident on a "step" potential function:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \qquad V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases}$$

Calculate the transmission coefficient $T = 1 - R$ as function of the energy, for the case that $E > V_0$.

Solution (Q4)

Before the step we have solutions of the form: $\psi(x \leq 0) = Ae^{ikx} + Be^{-ikx}$, and after the step: $\psi(x > 0) = Ce^{iqx}$. Here $k = \sqrt{2mE}/\hbar$ and $q = \sqrt{2m(E - V_0)}/\hbar$.

Applying boundary conditions of continuity of the wave function $\psi(x)$ and its derivative $\frac{d\psi(x)}{dx}$ in $x = 0$ gives the following equations:

$$\begin{cases} Ae^{ikx} + Be^{-ikx} = Ce^{iqx} \\ ikAe^{ikx} - ikBe^{-ikx} = iqCe^{iqx} \end{cases}$$

We will derive the transmission from the reflection coefficient R. To calculate the reflection coefficient $R = |B|^2/|A|^2$ we multiply the first equation by $-iq$ and add both equations together giving:

$$\begin{aligned} &(ik - iq)Ae^{ikx} - B(ik + iq)e^{-ikx} = 0 \\ &\Rightarrow (ik - iq)Ae^{ikx} = B(ik + iq)e^{-ikx} \\ &\Rightarrow \frac{B}{A} = \frac{(ik - iq)}{(ik + iq)}e^{i2kx} \\ &\Rightarrow R = \frac{|B|^2}{|A|^2} = \frac{|ik - iq|^2}{|ik + iq|^2}|e^{i2kx}|^2 = \frac{(k - q)^2}{(k + q)^2} = \frac{(k + q)^2 - 4qk}{(k + q)^2} \\ &\Rightarrow 1 - R = 1 - \frac{(k + q)^2 - 4qk}{(k + q)^2} = \frac{4qk}{(k + q)^2} = \frac{4\sqrt{E(E - V_0)}}{(\sqrt{E} + \sqrt{E - V_0})^2} \end{aligned}$$

Question 5: Order of projections of a state

Consider a system with the following orthonormal basis of eigenstates:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consider further the following two projection operators in this system:

- Projection operator to the second basis vector: $\hat{P}_2 = |2\rangle\langle 2|$
- Projection operator $\hat{P}_\alpha = |\alpha\rangle\langle\alpha|$ with $|\alpha\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(1/2 of points) Project the state $|1\rangle$ first onto $|\alpha\rangle$ and afterwards to $|2\rangle$, this means:

$$\hat{P}_2\hat{P}_\alpha|1\rangle = ?$$

(1/2 of points) Afterwards, swap the order and calculate again:

$$\hat{P}_{\alpha}\hat{P}_2|1\rangle = ?$$

Solution (Q5)

To calculate the projections we write them out and use the fact that $\{|1\rangle, |2\rangle\}$ are orthonormal:

$$\hat{P}_2\hat{P}_{\alpha}|1\rangle = |2\rangle\langle 2|\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\frac{1}{\sqrt{2}}(\langle 1| + \langle 2|)|1\rangle = \frac{1}{2}|2\rangle$$

When the order is swapped then the first projection on $|2\rangle$ gives the zero state:

$$\hat{P}_{\alpha}\hat{P}_2|1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\frac{1}{\sqrt{2}}(\langle 1| + \langle 2|)|2\rangle\langle 2||1\rangle = \hat{P}_{\alpha}\, 0|2\rangle = 0|\alpha\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Question 6: Commutators

Calculate the commutator:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A},$$

for the following operators in their matrix representation:

$$\hat{A} \longrightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \hat{B} \longrightarrow B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Solution (Q6)

The commutator is given by:

$$[\hat{A}\hat{B}] = AB - BA = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2i\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Question 7: The spectrum of eigenvalues in matrix formalism

(1/2 of points) Extract the eigenvalues for the following observable operator $\hat{Q}$ in a system described in a three-dimensional vector space:

$$Q|\alpha\rangle = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(1/2 of points) One of the eigenvalues is zero: Find the corresponding eigenstate and normalize it.

Hint: Eigenvalues λ_n are given by the characteristic equation, i.e., you need to find λ_n such that the determinant of the matrix: $\det(\lambda_n \mathbb{1} - Q) = 0$.

Remember that the determinant can be calculated from Laplace's determinant expansion:

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13} \\ &= (a_{22}a_{33} - a_{23}a_{32})a_{11} - \ldots \end{aligned}$$

Solution (Q7)

To find the eigenvalues we write out the characteristic equation $\det(\lambda_n \mathbb{1} - Q) = 0$:

$$\begin{aligned} 0 = \det(\lambda_n \mathbb{1} - Q) &= \det \begin{pmatrix} \lambda - 1 & 0 & 1 \\ 0 & \lambda + 1 & 0 \\ 1 & 0 & \lambda - 1 \end{pmatrix} \\ &= (\lambda - 1)^2(\lambda + 1) - (\lambda + 1) \\ &= \left((\lambda - 1)^2 - 1\right)(\lambda + 1) \\ &= \left(\lambda^2 - 2\lambda\right)(\lambda + 1) \\ &= \lambda(\lambda - 2)(\lambda + 1) \end{aligned}$$

Resulting in eigenvalues $\lambda_n = 0, -1, 2$. The eigenstates can be calculated by filling in the eigenvalues:

$$\boxed{\lambda = 0}: \qquad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow a = c \quad \& \quad b = 0 \qquad \Longrightarrow \quad |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Together with the other eigenvectors (not required for the answer):

$$|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad |-1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$