PHOT 301: Quantum Photonics

Midterm exam questions (retake)

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General information on the exam

Grading: This midterm exam will count for 10% of your total grade. Together with the projects which count for 40%, and the final exam that will account for 50%, your total grade for the course will be determined.

Exam type: The midterm exam consists of 7 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

Question 1: Wave functions and expectation values

Consider the following wave function defined on $x \in [0, 1]$:

$$\psi(x) = Ax(1-x)^2$$

with A a normalization constant.

(1/3) First calculate the normalization constant A of the wave function.
(2/3) Then calculate the expectation value for the position operator $\langle \hat{x} \rangle \in [0, 1]$.
(3/3) Afterwards calculate the expectation value for the momentum: $\langle \hat{p} \rangle = \left\langle -i\hbar \frac{d}{dx} \right\rangle$.

Hints: For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value for an operator $\hat{Q}$ is given by:

$$\langle \hat{Q} \rangle = \int_a^b \psi(x)^* \, \hat{Q} \, \psi(x) dx.$$

You also can make use of the following definite integral (here parameters n and m are integers):

$$\int_0^1 x^m \, (1-x)^n \, dx = \frac{m! \, n!}{(m+n+1)!},$$

where the factorial of a positive integer $j! = j \cdot (j-1) \cdot (j-2) \cdot \dots \cdot 2 \cdot 1$.

Question 2: Ladder operators in a harmonic oscillator

(1/1) Prove the following equality:

$$\left[\hat{a}_- \hat{a}_+ \hat{a}_+ + 2\hat{a}_- - \hat{a}_+\right] \psi_0(x) = \psi_1(x)$$

with $\psi_n(x)$ the eigenstates of the harmonic oscillator with Hamiltonian:

$$\hat{H} = \hbar\omega \left(\hat{a}_+ \hat{a}_- + \frac{1}{2} \right),$$

Hint: the commutator $[\hat{a}_-, \hat{a}_+] = \mathbb{1}$ and the ladder operators acting on an eigenstate $\psi_n(x)$:

$$\hat{a}_+ \psi_n(x) = \sqrt{n+1} \psi_{n+1}(x), \qquad \hat{a}_- \psi_n(x) = \sqrt{n} \psi_{n-1}(x)$$

Question 3: Oscillations of eigenstates

For an infinite square well with width L the solutions can be written in the form:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}, \quad \text{with } n = 1, 2, 3, \dots$$

Consider the wave function $\Psi(x,t)$, which has the following form:

$$\Psi(x,t) = A \left(\psi_1(x) e^{-iE_1 t/\hbar} + i \psi_2(x) e^{-iE_2 t/\hbar} \right).$$

(1/2) Find the normalization factor A for the wave function.
(2/2) Calculate the expectation value for the position $\langle \hat{x} \rangle$ as a function of time and show that it oscillates (in time) around the center of the well $L/2$.

Hints: For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value as function of time t of a time-independent operator $\hat{Q}$ is given by:

$$\langle \hat{Q} \rangle (t) = \int_a^b \Psi(x,t)^* \, \hat{Q} \, \Psi(x,t) dx.$$

You can further make use of the following definite integral:

$$\int_0^1 x \sin(\pi x) \sin(2\pi x) dx = \frac{8}{9\pi^2}.$$

Question 4: Bound states in a finite potential well

Consider the bound states in a 1D finite potential well,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \qquad V(x) = \begin{cases} 0 & -a < x < a \\ V_0 & x > a \text{ or } x < -a \end{cases}$$

Suppose the potential well height V_0 and the eigenenergy E_1 of the first eigenstate $\psi_1(x)$ is given by:

$$E_1 = \frac{\hbar^2}{2m}\frac{z^2}{a^2} = \frac{\hbar^2}{2m}\frac{\pi^2}{16a^2} = \frac{E_1^\infty}{4}, \qquad V_0 = \frac{\hbar^2}{2m}\frac{\pi^2}{8a^2} = \frac{E_1^\infty}{2},$$

where E_1^∞ is the ground state energy of the infinity well with the same width $L = 2a$

(1/1) Calculate the probability to be *outside the well* if the system is in the first eigenstate, that is, the probability you find the particle with $|x| > a$.

Hints: The wave function solution for the first eigenstate $\psi(x) = \psi_1(x)$ is given by:

$$\begin{cases} \psi_1(x < -a) = Be^{\kappa x} \\ \psi_1(-a < x < a) = D\cos(kx) \\ \psi_1(x > a) = Be^{-\kappa x} \end{cases},$$

with B and D constants and $k = \frac{\sqrt{2mE_1}}{\hbar}$ and $\kappa = \frac{\sqrt{2m(V_0 - E_1)}}{\hbar}$ extracted from the Schrodinger equation as usual.

You can further use the following indefinite integrals (anti-derivatives), where $b \in \mathbb{R}$ is a parameter:

$$\int \cos^2(bx) dx = \frac{2bx + \sin(2bx)}{4b}, \qquad \int e^{-bx} dx = -\frac{e^{-bx}}{b}$$

Question 5: Projection operators

Consider a system with the following orthonormal basis of eigenstates:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider further the projection operator $\hat{P}_\alpha = |\alpha\rangle\langle\alpha|$ with:

$$|\alpha\rangle = \frac{1}{\sqrt{5}}\left(2i|1\rangle + |2\rangle\right) = \frac{1}{\sqrt{5}}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

(1/1) Perform the following projection:

$$\hat{P}_\alpha(|1\rangle - |2\rangle) = ?$$

Question 6: Commutators

(1/1) Calculate the commutator:

$$[\hat{x}^2, \hat{p}^2] = \hat{x}^2\hat{p}^2 - \hat{p}^2\hat{x}^2,$$

with the position operator $\hat{x} = x$ and the momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$.

Question 7: The spectrum of eigenvalues in matrix formalism

(1/2) Extract the eigenvalues for the following observable operator $\hat{Q}$ in a system described in a two-dimensional vector space:

$$Q\,|\alpha\rangle = \begin{pmatrix} \sqrt{2} & 1+i \\ 1-i & -\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

(2/2) Afterwards, find the corresponding eigenstates and normalize them.

Hint: Eigenvalues λ_n are given by the characteristic equation, i.e., you need to find λ_n such that the determinant of the matrix: $\det(\lambda_n \mathbb{1} - Q) = 0$.

The determinant of a 2×2 matrix can be calculated as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$