PHOT 301: Quantum Photonics

Midterm exam questions

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General information on the exam

Grading: This midterm exam will count for 10% of your total grade. Together with the projects which count for 40%, and the final exam that will account for 50%, your total grade for the course will be determined.

Exam type: The midterm exam consists of 7 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

Question 1: Wave function in half-space

Consider the following wave function defined on $x \in [0, +\infty]$:

$$\psi(x) = Axe^{-\frac{\pi x^2}{2}}$$

with A the normalization constant.

- First calculate the normalization constant A of the wave function.
- Then calculate the expectation value for the position operator $\langle \hat{x} \rangle \in [0, +\infty]$.
- Afterwards calculate the variance: $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 - 2\langle x \rangle x + \langle x \rangle^2 \rangle$.

Hint: For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of an operator $\hat{Q}$ is given by:

$$\langle \hat{Q} \rangle = \int_a^b \psi(x)^* \, \hat{Q} \, \psi(x) dx.$$

You also can make use of the following definite integrals (here parameter $a > 0$):

$$\int_0^\infty e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{a}}, \qquad \int_0^\infty x e^{-ax^2} \, dx = \frac{1}{2a}, \qquad \int_0^\infty x^2 e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{4a^{3/2}},$$

$$\int_0^\infty x^3 e^{-ax^2} \, dx = \frac{1}{2a^2}, \qquad \int_0^\infty x^4 e^{-ax^2} \, dx = \frac{3\sqrt{\pi}}{8a^{5/2}}.$$

Question 2: Harmonic oscillator

Assume a particle in a harmonic oscillator potential is initially in the state:

$$\Psi(x, 0) = A \; (4\psi_0(x) + 3\psi_1(x))$$

with $\psi_n(x)$ the eigenstates of the harmonic oscillator.

- Find the normalization factor A of the wave function.
- Then show that the probability density function $|\Psi(x, t)|^2$ contains a time-dependent term which is proportional to $\cos(\omega t)$ by writing out its definition.

Hint: the eigenenergies corresponding to the the 1D harmonic oscillator (orthonormal) eigenstates: $\psi_n(x)$, are of the form:

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right),$$

and you can assume that the eigenstates themselves are real: $\psi_n(x)^* = \psi_n(x)$.

Question 3: Expansion in eigenstates

For an infinite square well with width L the solutions can be written in the form:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}, \quad \text{with } n = 1, 2, 3, \ldots$$

- Derive the coefficients c_1 and c_2 for the wave function $\psi(x) = c_1\psi_1 + c_2\psi_2$, if you know that the expectation value for the energy $\langle \hat{H} \rangle = 3E_1$.

Hint: For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of an operator $\hat{Q}$ is given by:

$$\langle \hat{Q} \rangle = \int_a^b \psi(x)^* \, \hat{Q} \, \psi(x) dx.$$

Question 4: Transmission through a step potential

Consider propagating waves in a 1D system coming from the left, incident on a "step" potential function:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \qquad V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases}$$

Calculate the transmission coefficient $T = 1 - R$ as function of the energy, for the case that $E > V_0$.

Question 5: Order of projections of a state

Consider a system with the following orthonormal basis of eigenstates:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consider further the following two projection operators in this system:

- Projection operator to the second basis vector: $\hat{P}_2 = |2\rangle\langle 2|$
- Projection operator $\hat{P}_\alpha = |\alpha\rangle\langle\alpha|$ with $|\alpha\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Project the state $|1\rangle$ first onto $|\alpha\rangle$ and afterwards to $|2\rangle$, this means:

$$\hat{P}_2 \hat{P}_\alpha |1\rangle = ?$$

Afterwards, swap the order and calculate again:

$$\hat{P}_\alpha \hat{P}_2 |1\rangle = ?$$

Question 6: Commutators

Calculate the commutator:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A},$$

for the following operators in their matrix representation:

$$\hat{A} \longrightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \hat{B} \longrightarrow B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Question 7: The spectrum of eigenvalues in matrix formalism

Extract the eigenvalues for the following observable operator $\hat{Q}$ in a system described in a three-dimensional vector space:

$$Q\,|\alpha\rangle = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

One of the eigenvalues is zero: Find the corresponding eigenstate and normalize it.

Hint: Eigenvalues λ_n are given by the characteristic equation, i.e., you need to find λ_n such that the determinant of the matrix: $\det(\lambda_n \mathbb{1} - Q) = 0$.

Remember that the determinant can be calculated from Laplace's determinant expansion:

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13} \\ &= (a_{22}a_{33} - a_{23}a_{32})a_{11} - \ldots \end{aligned}$$