PHOT 301: Quantum Photonics

Midterm exam example questions with solutions

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General information on the exam

Grading: This midterm exam will count for 10% of your total grade. Together with the projects which count for 40%, and the final exam that will account for 50%, your total grade for the course will be determined.

Exam type: The midterm exam consists of 7 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is one hour and 45 minutes.

Example exam questions

This document contains example questions for the midterm exam. The questions of the actual midterm exam will be different, but the style of the questions and covered topics should be similar.

The different topics of the exam questions cover:

1. The wave function, probability and statistical interpretation
2. Time-independent Schrodinger equation and the Hamiltonian
3. Eigenstates: expansion, completeness, orthonormality
4. Bound states and scattering
5. Dirac notation
6. Uncertainty relations
7. Matrix formalism for finite basis

Please fill in all questions listed below.

Question 1: Sinc wave function

Calculate the expectation value $\langle \hat{Q} \rangle$ of the operator $\hat{Q} = x\,\frac{d}{dx}$ for wave function $\psi(x) = A\,\mathrm{sinc}(x) = A\,\frac{\sin(x)}{x}$ defined in the interval $x \in [-\infty, +\infty]$. First calculate the normalization constant A of the wave function.

Hint: For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of an operator $\hat{Q}$ is given by:

$$\langle \hat{Q} \rangle = \int_a^b \psi(x)^* \,\hat{Q}\, \psi(x) dx.$$

You also make use of the following definite integrals:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi, \qquad \int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 dx = \pi, \qquad \int_{-\infty}^{\infty} \frac{\cos(x)\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

Solution (Q1)

The normalization factor $A = 1/\sqrt{\pi}$ as can be seen from:

$$1 = |A|^2 \int \frac{\sin^2(x)}{x^2}\, dx$$

The expectation value $\langle \hat{Q} \rangle$ can be calculated as follows:

$$\begin{aligned}
\langle \hat{Q} \rangle = \langle x\frac{d}{dx} \rangle = |A|^2 \int \frac{\sin(x)}{x} \left(x\frac{d}{dx} \right) \frac{\sin(x)}{x}\, dx \\
\frac{1}{\pi} \int \frac{\sin(x)}{x}\, x \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right) dx \\
\frac{1}{\pi} \int \left(\frac{\sin(x)\cos(x)}{x} - \frac{\sin^2(x)}{x^2} \right) dx \\
\frac{1}{\pi} \left(\frac{\pi}{2} - \pi \right) = -\frac{1}{2}
\end{aligned}$$

Question 2: Ladder operators

The creation and annihilation ladder operators $\hat{a}_+$ and $\hat{a}_-$, acting on the eigenstates of the 1D harmonic oscillator: $\psi_n(x)$, have the following properties:

$$\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1},$$
$$\hat{a}_-\psi_n = \sqrt{n}\psi_{n-1}.$$

Derive from these properties that the two products of these operators $(\hat{a}_+\hat{a}_-)$ and $(\hat{a}_-\hat{a}_+)$ have the following eigenvalues:

$$\hat{a}_+\hat{a}_-\psi_n = n\psi_n,$$
$$\hat{a}_-\hat{a}_+\psi_n = (n+1)\psi_n.$$

Solution (Q2)

$$\hat{a}_+\hat{a}_-\psi_n = \hat{a}_+\sqrt{n}\psi_{n-1} = \sqrt{n}\hat{a}_+\psi_{n-1} = \sqrt{n}\sqrt{n}\psi_n = n\psi_n,$$
$$\hat{a}_-\hat{a}_+\psi_n = \hat{a}_-\sqrt{n+1}\psi_{n+1} = \sqrt{n+1}\hat{a}_-\psi_{n+1} = \sqrt{n+1}\sqrt{n+1}\psi_n = (n+1)\,\psi_n.$$

Question 3: Expansion in eigenstates

For an infinite square well with width L the solutions can be written in the form:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\hbar^2\pi^2n^2}{2mL^2}, \quad \text{with } n = 1, 2, 3, \ldots$$

Assume that the system is initially at time $t = 0$ in the state with wave function $\Psi(x,0) = A\,(\psi_1(x) - 2\,i\psi_2(x))$ with A a normalization constant.

- Calculate the value of the normalization constant A.
- Calculate the expectation value for the energy: $\langle\hat{H}\rangle$.

Solution (Q3)

The normalization factor A of the wave function $A\,(\psi_1(x) - 2\,i\psi_2(x))$ can be calculated as follows:

$$
\begin{aligned}
1 = \int |\psi(x)|^2 dx &= |A|^2 \int \left(\psi_1(x)^* + 2i\psi_2(x)^*\right)\left(\psi_1(x) - 2i\psi_2(x)\right) dx \\
&= |A|^2 \int |\psi_1(x)|^2 + 4|\psi_2(x)|^2 \, dx \\
&= |A|^2(1+4) = |A|^2\, 5 \\
\Longrightarrow & |A| = 1/\sqrt{5}
\end{aligned}
$$

The expectation value of the energy $\langle \hat{H} \rangle$:

$$
\begin{aligned}
\langle \hat{H} \rangle = \int \psi(x)^* \hat{H} \psi(x)\, dx &= |A|^2 \int \left(\psi_1(x)^* + 2i\psi_2(x)^*\right) \hat{H} \left(\psi_1(x) - 2i\psi_2(x)\right) dx \\
&= |A|^2 \int \left(\psi_1(x)^* + 2i\psi_2(x)^*\right)\left(E_1\psi_1(x) - 2iE_2\psi_2(x)\right) dx \\
&= \frac{1}{5} \int E_1|\psi_1(x)|^2 + 4E_2|\psi_2(x)|^2 \, dx \\
&= \frac{E_1}{5} + \frac{4E_2}{5} = \frac{\hbar^2\pi^2}{2mL^2}\frac{17}{5}
\end{aligned}
$$

Question 4: Reflection at a step potential

Consider a "step" potential function:

$$
V(x) = \begin{cases} 0 & x \le 0 \\ V_0 & x > 0 \end{cases}
$$

Calculate the reflection coefficient R for the case that $E < V_0$.

Solution (Q4)

Before the step we have solutions of the form: $\psi(x \le 0) = Ae^{ikx} + Be^{-ikx}$, and behind the step: $\psi(x > 0) = Ce^{-\kappa x}$. Here $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(V_0 - E)}/\hbar$.

Applying boundary conditions of continuity of the $\psi(x)$ and its derivative $\frac{d\psi(x)}{dx}$ in $x = 0$ gives the following equations:

$$
\begin{cases} Ae^{ikx} + Be^{-ikx} = Ce^{-\kappa x} \\ ikAe^{ikx} - ikBe^{-ikx} = -\kappa Ce^{-\kappa x} \end{cases}
$$

To calculate the reflection coefficient $R = |B|^2/|A|^2$ we multiply the first equation by κ and add both equations together giving:

$$
\begin{aligned}
(ik+\kappa)Ae^{ikx} + B(-ik+\kappa)e^{-ikx} = 0 \\
\Rightarrow (ik+\kappa)Ae^{ikx} = B(ik-\kappa)e^{-ikx} \\
\Rightarrow B/A = \frac{(ik+\kappa)}{(ik-\kappa)}e^{i2kx} \\
\Rightarrow R = |B|^2/|A|^2 = \frac{|ik+\kappa|^2}{|ik-\kappa|^2}|e^{i2kx}|^2 = \frac{k^2+\kappa^2}{k^2+\kappa^2} = 1
\end{aligned}
$$

Question 5: Projection of a state

Consider a system with the following orthonormal basis of eigenstates:

$$
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Within this system, project the state $|\beta\rangle$ onto $|\alpha\rangle$ by applying the projection operator $\hat{P}_\alpha$:

$$
\hat{P}_\alpha|\beta\rangle = |\alpha\rangle\langle\alpha|\beta\rangle = \langle\alpha|\beta\rangle\,|\alpha\rangle, \qquad \text{where} \quad |\alpha\rangle = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \qquad |\beta\rangle = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}
$$

Solution (Q5)

Writing out the projector operator in matrix notation:

$$
\hat{P}_\alpha|\beta\rangle = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -i & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = (2-i) \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = (2-i)\,|\alpha\rangle
$$

Question 6: Commutators

Proof that the commutator between the third power of the position operator $\hat{x}^3$ and momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$ is given by:

$$
[\hat{x}^3, \hat{p}] = \hat{x}^3\hat{p} - \hat{p}\hat{x}^3 = i\hbar 3x^2 \mathbb{1}.
$$

Solution (Q6)

To calculated the commutator we apply it to a "test" function $f(x)$

$$
\begin{aligned}
[\hat{x}^3, \hat{p}]f(x) &= x^3\hat{p}f(x) - \hat{p}x^3 f(x) \\
&= (-i\hbar)x^3\frac{d}{dx}f(x) - (-i\hbar)\frac{d}{dx}(x^3 f(x)) \\
&= (-i\hbar)x^3\frac{df(x)}{dx} - (-i\hbar)(3x^2 f(x) + x^3\frac{df}{dx}) \\
&= i\hbar 3x^2 \mathbb{1}
\end{aligned}
$$

Question 7: The spectrum of eigenvalues in matrix formalism

Extract the eigenvalues for the following observable operator $\hat{Q} \longrightarrow \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$ in a system described in a two-dimensional vector space:

$$
Q|\alpha\rangle = \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}
$$

Then find corresponding eigenstates and normalize them.

Hint: Eigenvalues λ_n are given by the characteristic equation, i.e., you need to find λ_n such that the determinant of the matrix: $\det(\lambda_n \mathbb{1} - Q) = 0$.

Solution (Q7)

The characteristic equation allows us to calculate the eigenvalues λ_n as follows:

$$
\begin{aligned}
\det\begin{pmatrix} \lambda - 1 & -i \\ i & \lambda + 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= (\lambda - 1)(\lambda + 1) - 1 = 0 \\
&\Longrightarrow \lambda^2 = 2 \Longrightarrow \lambda = \pm\sqrt{2}
\end{aligned}
$$

To obtain the corresponding eigenstates we fill in the eigenvalues in the equation:

$$
\begin{pmatrix} (1 \mp \sqrt{2}) & i \\ -i & -1 \mp \sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

The first equation is $(1 \mp \sqrt{2})a = -ib$ which leads to:

$$
\frac{b}{a} = i(1 - \mp\sqrt{2}) \quad \Rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ i(1 \mp \sqrt{2}) \end{pmatrix}
$$

To normalize the eigenstates we verify the inner product $\langle\alpha|\alpha\rangle = \|\alpha\|^2$

$$\langle\alpha|\alpha\rangle = \begin{pmatrix} 1 & -i(1 \mp \sqrt{2}) \end{pmatrix} \begin{pmatrix} 1 \\ i(1 \mp \sqrt{2}) \end{pmatrix} = 1 + (1 \mp \sqrt{2})^2 = 4 \mp 2\sqrt{2}$$

Resulting eigenstates are given by:

$$\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}\sqrt{2 \mp \sqrt{2}}} \begin{pmatrix} 1 \\ i(1 \mp \sqrt{2}) \end{pmatrix}$$