PHOT 301: Quantum Photonics

Final exam questions

Michaël Barbier, Fall semester (2024-2025)

General information on the exam

Grading: This final exam will count for 50% of your total grade. Together with the projects which count for 40%, and the midterm exam that counts for 10%, your total grade for the course will be determined.

Exam type: The final exam consists of 8 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam. The last page of the exam contains some formulas which can be used to help solve the problems.

The duration of the final exam is 3 hours.

Exam questions

Please answer all questions listed below. Each of the questions is valued equally in the score calculation of the exam. If a question contains multiple parts, each of the parts is valued equally within the score for the question.

Please tell if any question is unclear or ambiguous.

Question 1: Wave functions and expectation values

Consider the following 1D wave function defined with $x \in [0, 1]$:

$$\psi(x) = A\,(x - 1)\,x^2$$

with A a normalization constant.

(1/2) First calculate the normalization constant A of the wave function.
(2/2) Then calculate the expectation value for the kinetic energy $\frac{1}{2m}\langle \hat{p}^2 \rangle = -\frac{\hbar^2}{2m}\left\langle \frac{d^2}{dx^2} \right\rangle$.

Solution (Q1)

The normalization factor $A = \sqrt{105}$ as can be seen from:

$$
\begin{aligned}
1 = \int_0^1 |\psi(x)|^2 dx &= |A|^2 \int_0^1 (x-1)^2 x^4 \, dx \\
&= |A|^2 \int_0^1 (x^6 - 2x^5 + x^4) \, dx \\
&= |A|^2 \left(\frac{x^7}{7} - \frac{x^6}{3} + \frac{x^5}{5} \right) \Big|_0^1 \\
&= |A|^2 \left(\frac{1}{7} - \frac{1}{3} + \frac{1}{5} \right) = \frac{|A|^2}{105}
\end{aligned}
$$

The expectation value for the kinetic energy $\langle \hat{p}^2 \rangle / 2m$ can be derived as:

$$
\begin{aligned}
\frac{1}{2m} \langle \hat{p}^2 \rangle &= -105 \frac{\hbar^2}{2m} \int_0^1 (x-1)x^2 \frac{d^2}{dx^2} [(x-1)x^2] \, dx \\
&= -105 \frac{\hbar^2}{2m} \int_0^1 (x-1)x^2 \, [6x - 2] \\
&= -105 \frac{\hbar^2}{2m} \int_0^1 (6x^4 - 8x^3 + 2x^2) \, dx \\
&= -105 \frac{\hbar^2}{2m} \left(\frac{6}{5} x^5 - \frac{8}{4} x^4 + \frac{2}{3} x^3 \right) \Big|_0^1 \\
&= -105 \frac{\hbar^2}{m} \left(\frac{3}{5} - 1 + \frac{1}{3} \right) = 7 \frac{\hbar^2}{m}
\end{aligned}
$$

Question 2: Orbital eigenstate probability current

Consider the 2P1 orbital with eigenstate represented by the wave function $\psi_{211}(\vec{r})$:

$$
\psi_{211}(\vec{r}) = R_{21}(r) \, Y_{11}(\theta, \phi) = -\frac{1}{8\sqrt{\pi a_0^3}} \frac{r}{a_0} e^{-\frac{r}{a_0}} \sin(\theta) \, e^{i\phi}
$$

(1/1) calculate the ϕ component of the probability current density $\vec{j}$ which is given by the formula:

$$
\vec{j} = (j_r, j_\theta, j_\phi) = -\frac{i\hbar}{2m} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right) = \frac{\hbar}{m} \Im \left\{ \psi^* \nabla \psi \right\}
$$

where $\Im\{\dots\}$ represents the imaginary part, and the gradient of a function f, that is, ∇f in spherical coordinates is given by

$$\nabla f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

Hereby you can assume that ϕ component j_ϕ is the only nonzero component (you don't need to prove that).

Solution (Q2)

$$\begin{aligned}
j_\phi &= \frac{\hbar}{m}\,\Im\left\{\psi^* \nabla_\phi \psi\right\} = \frac{\hbar}{m}\Im\left\{\frac{1}{r\sin(\theta)}\,\psi\,\frac{\partial \psi}{\partial \phi}\right\} \\
&= \frac{1}{8^2\pi a_0^5}\frac{\hbar}{m}\Im\left\{\frac{1}{r\sin(\theta)}\,re^{-\frac{r}{2a_0}}\sin(\theta)\,e^{-i\phi}\,\frac{\partial}{\partial \phi}\left[re^{-\frac{r}{2a_0}}\sin(\theta)\,e^{i\phi}\right]\right\} \\
&= \frac{1}{8^2\pi a_0^5}\,r\,e^{-\frac{r}{a_0}}\sin(\theta)\,\frac{\hbar}{m}\,\Im\left\{e^{-i\phi}\,\frac{\partial}{\partial \phi}\left[e^{i\phi}\right]\right\} \\
&= \frac{1}{8^2\pi a_0^5}\,r\,e^{-\frac{r}{a_0}}\sin(\theta)\,\frac{\hbar}{m}\,\Im\left\{i\right\} \\
&= \frac{\hbar}{8^2 m\pi a_0^5}\,r\,e^{-\frac{r}{a_0}}\sin(\theta)
\end{aligned}$$

which is nonzero and rotating around the center.

Question 3: Oscillations of eigenstates

Consider the following superpositon of 1D harmonic oscillator eigenstates with time-dependent factors:

$$\Psi(x,t) = \frac{1}{\sqrt{5}}\left(\psi_0\,e^{-i\omega t/2} + 2\psi_1\,e^{-i3\omega t/2}\right)$$

where the eigenstates can be written as:

$$\psi_0 = \alpha\,e^{-\beta^2 x^2/2}, \qquad \psi_1 = \sqrt{2}\alpha\beta\,x\,e^{-\beta^2 x^2/2}, \quad \text{with} \quad \alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}, \quad \beta = \sqrt{\frac{m\omega}{\hbar}}$$

(1/1) Calculate the expectation value for the position $\langle \hat{x} \rangle$ as a function of time and show that it oscillates (in time) around zero.

Solution (Q3)

$$
\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x\,\Psi^*\,\Psi\,dx \\
&= \frac{1}{5}\int_{-\infty}^{\infty} x\,\left(\psi_0\,e^{i\omega t/2}+\psi_1\,e^{i3\omega t/2}\right)\left(\psi_0\,e^{-i\omega t/2}+\psi_1\,e^{-i3\omega t/2}\right)\,dx \\
&= \frac{1}{5}\int_{-\infty}^{\infty} x\,\left(\psi_0^2+4\,\psi_1^2+4\cos(\omega t)\psi_0\psi_1\right)\,dx \\
&= \frac{1}{5}\int_{-\infty}^{\infty} \left(x\,\psi_0^2+4x\,\psi_1^2+4x\,\cos(\omega t)\psi_0\psi_1\right)\,dx \\
&= 0+0+\frac{1}{5}\int_{-\infty}^{\infty} 4x\,\cos(\omega t)\psi_0\psi_1\,dx \\
&= \frac{\sqrt{2}\,4}{5}\,\cos(\omega t)\alpha^2\beta\int_{-\infty}^{\infty} x^2\,e^{-\beta^2x^2}\,dx \\
&= \frac{\sqrt{2}\,4}{5}\,\cos(\omega t)\alpha^2\beta\frac{\sqrt{\pi}}{2\beta^3} = \frac{\sqrt{2}\,4}{5}\,\cos(\omega t)\alpha^2\frac{\sqrt{\pi}}{2\beta^2} = \frac{\sqrt{2}\,2}{5}\sqrt{\frac{\hbar}{m\omega}}\,\cos(\omega t)
\end{aligned}
$$

which is oscillating in time with frequency $\omega/2\pi$ and displacement $x = \pm\frac{\sqrt{2}\,2}{5}\sqrt{\frac{\hbar}{m\omega}}$ close to and proportional with the classical turning points at $x = \pm\sqrt{\frac{\hbar}{m\omega}}$.

Question 4: Perturbation: two state system

Consider a two-state system with following Hamiltonian:

$$
\hat{H} = \hat{H}_0 + \hat{H}_p = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} + \begin{pmatrix} 0 & -i\varepsilon \\ i\varepsilon & 0 \end{pmatrix}
$$

with Δ and ε positive real numbers, and assume $\varepsilon \ll \Delta$. The unperturbed Hamiltonian $\hat{H}_0$ has eigenenergies $E_m^{(0)}$ and $|\psi_m^{(0)}\rangle$, that is:

$$
|\psi_1^{(0)}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E_1 = \Delta, \quad \text{and} \quad |\psi_2^{(0)}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E_2 = -\Delta
$$

(1/3) Prove that the energy does not change within first order perturbation theory approximation. That is, prove that $\langle\psi_m^{(0)}|\hat{H}_p|\psi_m^{(0)}\rangle = 0$ for both eigenstates.
(2/3) Calculate the perturbed eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ within first order perturbation approximation.
(3/3) Normalize the perturbed eigenstates.

Hint: The perturbed eigenenergies E_m and eigenstates ψ_m up to first order in perturbation theory are given by:

$$E_m = E_m^{(0)} + \langle \psi_m^{(0)} | \hat{H}_p | \psi_m^{(0)} \rangle$$

$$|\psi_m\rangle = |\psi_m^{(0)}\rangle + \sum_{n \neq m} \frac{\langle \psi_n^{(0)} | \hat{H}_p | \psi_m^{(0)} \rangle}{E_m^{(0)} - E_n^{(0)}} |\psi_n^{(0)}\rangle$$

Solution (Q4)

(1/3) The energy up to first order perturbation for the ground state is given by:

$$E_1 = E_1^{(0)} + \langle \psi_1^{(0)} | \hat{H}_p | \psi_1^{(0)} \rangle = \Delta + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i\varepsilon \\ i\varepsilon & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Delta + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ i\varepsilon \end{pmatrix} = \Delta + 0,$$

$$E_2 = E_2^{(0)} + \langle \psi_2^{(0)} | \hat{H}_p | \psi_2^{(0)} \rangle = -\Delta + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i\varepsilon \\ i\varepsilon & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\Delta + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -i\varepsilon \\ 0 \end{pmatrix} = -\Delta + 0,$$

where the contribution due to first order perturbation is zero.

(2/3) The perturbed eigenstates are given by:

$$\begin{aligned} |\psi_1\rangle &= |\psi_1^{(0)}\rangle + \frac{\langle \psi_2^{(0)} | \hat{H}_p | \psi_1^{(0)} \rangle}{E_1^{(0)} - E_2^{(0)}} |\psi_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2\Delta} \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i\varepsilon \\ i\varepsilon & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i\varepsilon}{2\Delta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\varepsilon}{2\Delta} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} |\psi_2\rangle &= |\psi_2^{(0)}\rangle + \frac{\langle \psi_1^{(0)} | \hat{H}_p | \psi_2^{(0)} \rangle}{E_2^{(0)} - E_1^{(0)}} |\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{-2\Delta} \left[\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i\varepsilon \\ i\varepsilon & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{i\varepsilon}{2\Delta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{i\varepsilon}{2\Delta} \\ 1 \end{pmatrix} \end{aligned}$$

(3/3) To normalize the perturbed eigenstates we assume a normalization factor A and set the norm squared equal to one:

$$1 = \langle \psi_1 | \psi_1 \rangle = |A_1|^2 \begin{pmatrix} 1 & \frac{-i\varepsilon}{2\Delta} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{i\varepsilon}{2\Delta} \end{pmatrix} = |A_1|^2 \left(1 + \frac{\varepsilon^2}{4\Delta^2} \right),$$

$$1 = \langle \psi_2 | \psi_2 \rangle = |A_2|^2 \begin{pmatrix} \frac{-i\varepsilon}{2\Delta} & 1 \end{pmatrix} \begin{pmatrix} \frac{i\varepsilon}{2\Delta} \\ 1 \end{pmatrix} = |A_2|^2 \left(\frac{\varepsilon^2}{4\Delta^2} + 1 \right).$$

This results in normalization factor $A_1 = A_2 = A = \left(\sqrt{1 + \frac{\varepsilon^2}{4\Delta^2}}\right)^{-1}$.

Question 5: Operator in a three-state system

Consider a three-state system with following eigenenergies and corresponding eigenstates:

$$E_1 = \Delta, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad E_2 = 2\Delta, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad E_3 = 3\Delta, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where Δ is some energy quantity.

Consider further the operator $\hat{Q}$ given in matrix-formalism by:

$$\hat{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 1 \end{pmatrix}$$

Assume the system is initially in the state $|\alpha\rangle = \frac{1}{\sqrt{5}}\,(\,2|1\rangle + |3\rangle\,) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

(1/3) What is the energy of the state $|\alpha\rangle$?
(2/3) Perform the operation: $\hat{Q}\,|\alpha\rangle = ?$
(3/3) What is the energy of the state after the operation?

Solution (Q5)

(1/3) The energy E of a state expanded in eigenstates of the system $|\psi\rangle = \sum_n c_n |\psi_n\rangle$ can be written as:

$$E = \sum_n |c_n|^2 E_n,$$

which for the state $|\alpha\rangle$ expanded in the three eigenstates $|n\rangle$ with $E_n = n\Delta$ becomes:

$$E = \left(\frac{2}{\sqrt{5}}\right)^2 E_1 + \left(\frac{1}{\sqrt{5}}\right)^2 E_3 = \frac{4\Delta}{5} + \frac{3\Delta}{5} = \frac{7}{5}\Delta$$

(2/3) Operating with $\hat{Q}$ gives:

$$\hat{Q}|\alpha\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 2+i \\ 0 \\ 1-2i \end{pmatrix}$$
$$= \frac{2+i}{\sqrt{10}}|1\rangle + \frac{1-2i}{\sqrt{10}}|3\rangle$$

(3/3) The energy can be calculated in the same manner as in the first sub-question:

$$E = \left|\frac{2+i}{\sqrt{10}}\right|^2 E_1 + \left|\frac{1-2i}{\sqrt{10}}\right|^2 E_3 = \frac{\Delta}{2} + \frac{3\Delta}{2} = 2\Delta$$

Question 6: Spin in a magnetic field

A B-field is oriented along the diagonal between the x and z-axis: $\vec{B} = \frac{B}{\sqrt{2}}(1, 0, 1)$. That is, consider the following Schrodinger (Pauli) equation:

$$\mu_B \frac{B}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_u \\ c_d \end{pmatrix} = E\begin{pmatrix} c_u \\ c_d \end{pmatrix}$$

where $\mu_B = \frac{e\hbar}{2m_0}$ is the Bohr magneton.

(1/1) Calculate the eigenenergies and corresponding eigenstates of the system.

Solution (Q6)

We solve the eigenvalue equation first for the eigenenergies, to simplify the notation we define $b \equiv \mu_B \frac{B}{\sqrt{2}}$. The eigenvalue equation can then be written:

$$\begin{pmatrix} b & b \\ b & -b \end{pmatrix}\begin{pmatrix} c_u \\ c_d \end{pmatrix} = E\begin{pmatrix} c_u \\ c_d \end{pmatrix}$$

The solution of this equation is given by the roots of the determinant $\det(\hat{H} - E\mathbb{1})$:

$$\det(\hat{H} - E\mathbb{1}) = \det\begin{pmatrix} b-E & b \\ b & -b-E \end{pmatrix} = (E-b)(E+b) - b^2 = E^2 - 2b^2$$

$$\Rightarrow \qquad E_{\pm} = \pm\sqrt{2}b = \pm\mu_B B$$

The corresponding eigenstates $|\psi_{\pm}\rangle$ can be derived by filling in the eigenenergies in the equation:

$$\begin{pmatrix} b & b \\ b & -b \end{pmatrix}\begin{pmatrix} c_u \\ c_d \end{pmatrix} = E_{\pm}\begin{pmatrix} c_u \\ c_d \end{pmatrix},$$

leading to

$$\begin{cases} bc_u + bc_d = \pm\sqrt{2}bc_u \\ bc_u - bc_d = \pm\sqrt{2}bc_d \end{cases} \Rightarrow \quad \begin{cases} c_u + c_d = \pm\sqrt{2}c_u \\ c_u - c_d = \pm\sqrt{2}c_d \end{cases} \Rightarrow \quad (1 \pm \sqrt{2}) = \frac{c_u}{c_d}$$

The eigenstates are then given by:

$$|\psi_\pm\rangle = A_\pm \begin{pmatrix} (1 \pm \sqrt{2}) \\ 1 \end{pmatrix},$$

where $A_\pm$ is a normalization factor. We can calculate the normalization factor by setting the squared norm to one:

$$1 = \langle\psi_\pm|\psi_\pm\rangle = |A_\pm|^2 \begin{pmatrix} (1 \pm \sqrt{2}) & 1 \end{pmatrix} \begin{pmatrix} (1 \pm \sqrt{2}) \\ 1 \end{pmatrix} = |A_\pm|^2[(1\pm\sqrt{2})^2+1] = |A_\pm|^2(4\pm2\sqrt{2}),$$

resulting in $A_\pm = (4 \pm 2\sqrt{2})^{-1/2}$.

Question 7: Periodic systems

Assume a 1D periodic system of equi-distant potential barriers. A single unit cell has length L and contains a single potential barrier. The relation between the coefficients (A, B) before the barrier at $x = 0$ and (C, D) after the barrier at $x = L$ can be written as:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

According to the Bloch theorem we can write

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} e^{i\beta L} & 0 \\ 0 & e^{i\beta L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

with β the Bloch wave vector which provides the slowly varying phase of the envelope function. Combining the above two conditions leads to a single matrix equation.

(1/1): Derive the characteristic equation (which defines the band structure $E(\beta)$), by combining the above equations and solving. Split the resulting equation into a real and imaginary part of the equation by making use of the fact that the determinant of the transfer-matrix equals one: $\det(T) = 1$.

Solution (Q7)

If we bring $e^{i\beta L}$ to the left-hand-side of the second (Bloch) condition, then we can express everything in coefficients C and D:

$$\left[\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} - \begin{pmatrix} e^{-i\beta L} & 0 \\ 0 & e^{-i\beta L} \end{pmatrix}\right] \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant of the total matrix on the left-hand-side should be zero to fullfil the conditions:

$$\det \begin{pmatrix} t_{11} - e^{-i\beta L} & t_{12} \\ t_{21} & t_{22} - e^{-i\beta L} \end{pmatrix} = 0$$

$$\Rightarrow \quad (t_{11} - e^{-i\beta L})(t_{22} - e^{-i\beta L}) - t_{12}t_{21} = 0$$

$$\Rightarrow \quad (t_{11}t_{22} - t_{12}t_{21}) - (t_{11} + t_{22})\, e^{-i\beta L} + e^{-i2\beta L} = 0$$

Multiplying the equation by $e^{i\beta L}$ and using the fact that $t_{11}t_{22} - t_{12}t_{21} = \det(T) = 1$ we obtain:

$$e^{i\beta L} - (t_{11} + t_{22}) + e^{-i\beta L} = 0$$

$$\Rightarrow \quad 2\cos(\beta L) = t_{11} + t_{22}$$

The last equation can now be split in real and imaginary parts:

$$\begin{cases} \cos(\beta L) = \frac{1}{2}\Re\{t_{11} + t_{22}\} \\ 0 = \Im\{t_{11} + t_{22}\} \end{cases}$$

Thereby we see that the real part of the trace of the transfer matrix determines the relation with the Bloch wave vector.

Question 8: two fermions in a harmonic oscillator

Consider a 1D harmonic oscillator with two identical fermionic particles (both have same spin). Assume that the particles are not interacting and ignore exchange energy. Consider further that one of the single-particle functions is in the ground-state and one is in the first excited state. For two fermions the wave function is the anti-symmetric sum (Slater determinant) of the single-particle functions:

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)),$$

where x_1 is the x-coordinate of particle 1 and x_2 is the x-coordinate of particle 2.

(1/1) Prove that this wave function is normalized, this means:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x_1, x_2)|^2 \, dx_1 \, dx_2 = 1$$

For this you can make use of the fact that the single-particle eigenstates $\psi_n(x)$ are orthonormal and real-valued.

Solution (Q8)

We can show that the wave function is normalized as follows:

$$\begin{aligned}
\int |\psi|^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x_1, x_2)\psi(x_1, x_2) \, dx_1 \, dx_2 \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)] \times [\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)] \, dx_1 \, dx_2 \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\psi_0^2(x_1)\psi_1^2(x_2) + \psi_0^2(x_2)\psi_1^2(x_1) - 2\,\psi_0(x_1)\psi_1(x_2)\psi_0(x_2)\psi_1(x_1)] \, dx_1 \, dx_2 \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi_0^2(x_1)dx_1 \int_{-\infty}^{\infty} \psi_1^2(x_2)dx_2 + \frac{1}{2}\int_{-\infty}^{\infty} \psi_1^2(x_1)dx_1 \int_{-\infty}^{\infty} \psi_0^2(x_2)dx_2 \\
&\quad - \int_{-\infty}^{\infty} \psi_0(x_1)\psi_1(x_1)dx_1 \int_{-\infty}^{\infty} \psi_0(x_2)\psi_1(x_2) \, dx_2 \\
&= \frac{1}{2} + \frac{1}{2} - 0 = 1
\end{aligned}$$

Formulas

In the following formulas parameters n, m are integers and $0 < a \in \mathbb{R}$ and $b \in \mathbb{R}_0$:

Anti-derivatives (indefinite integrals)

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\arctan(x) + \frac{x}{x^2+1}\right)$$
$$\int \frac{x}{(x^2+1)^2}\,dx = -\frac{1}{2}\frac{1}{x^2+1}$$
$$\int \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\arctan(x) - \frac{x}{x^2+1}\right)$$
$$\int \frac{x^3}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\frac{1}{x^2+1} + \log(x^2+1)\right)$$

$$\int \cos^n(ax)\sin(ax)\,dx = -\frac{1}{a(n+1)}\cos^{n+1}(ax)$$
$$\int \cos(ax)\sin^n(ax)\,dx = \frac{1}{a(n+1)}\sin^{n+1}(ax)$$

Definite integrals

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \frac{1}{2}\delta_{mn}$$
$$\int_0^1 x\sin^2(m\pi x)\,dx = \frac{1}{4}$$
$$\int_0^1 x^2\sin^2(m\pi x)\,dx = \frac{1}{6} - \frac{1}{4\pi^2 m^2}$$
$$\int_0^1 x\sin(\pi x)\sin(3\pi x)\,dx = 0$$
$$\int_0^1 x^2\sin(\pi x)\sin(3\pi x)\,dx = \frac{3}{16\pi^2}$$
$$\int_0^1 x^3\sin(\pi x)\sin(3\pi x)\,dx = \frac{9}{32\pi^2}$$

Definite integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$
$$\int_0^\infty e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$
$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$
$$\int_0^\infty x^4 e^{-ax^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$

$$\int_{-\infty}^\infty e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{\sqrt{a}}$$
$$\int_{-\infty}^\infty x^2 e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$
$$\int_{-\infty}^\infty x^4 e^{-ax^2}\,dx = \frac{3\sqrt{\pi}}{4a^{5/2}}$$

Integration in spherical coordinates:

$$\int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \int_{-\infty}^\infty dz\, f(x,y,z) = \int_0^\infty d\rho \int_0^\pi d\theta \int_0^{2\pi} d\phi\, \rho^2 \sin\theta\; F(\rho,\theta,\phi)$$

where volume element $dx\,dy\,dz = \rho^2 \sin\theta\, d\theta\, d\phi\, d\rho$

$x = \rho\sin(\theta)\cos(\phi),\ y = \rho\sin(\theta)\sin(\phi),\ z = \rho\cos(\theta)$