PHOT 301: Quantum Photonics

Final exam: questions & solutions

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Exam questions

Grading: The final exam counts for 40% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the final exam is 3 hours.

This document contains both the problems and their solutions.

Question 1: Wave functions

Consider the time-independent wave function $\psi(x)$ defined with $x \in \mathbb{R}$:

$$\psi(x) = Ax\, e^{-|x|}$$

(a) First calculate the normalization constant A of the wave function.
(b) Then calculate the probability to find the particle in interval $[-1, 1]$.

Solution (Q1)

(a) The normalization constant A can be found by setting the total probability equal to one:

$$1 = \int_{-\infty}^{+\infty} |\psi(x)|^2\, dx = \int_{-\infty}^{+\infty} |Ax\, e^{-|x|}|^2\, dx = |A|^2 \int_{-\infty}^{+\infty} x^2\, e^{-2|x|}\, dx$$
$$= 2|A|^2 \int_0^{+\infty} x^2\, e^{-2x}\, dx = 2|A|^2 \frac{2!}{2^3} = \frac{1}{2}\, |A|^2$$

Therefore $A = \sqrt{2}$ when chosen real-valued and positive.

(b) The probability to find the particle in interval $[-1,1]$ is given by:

$$\begin{aligned} P(x \in [-1,1]) &= \int_{-1}^{1} |\psi(x)|^2 \, dx \\ &= |A|^2 \int_{-1}^{1} x^2 \, e^{-2|x|} \, dx \\ &= 2|A|^2 \int_{0}^{1} x^2 \, e^{-2x} \, dx \\ &= -4 \left[\left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4} \right) e^{-2x} \right] \Bigg|_0^1 \\ &= 1 - 5e^{-2} \approx 1 - 0.68 \approx 0.32, \end{aligned}$$

which is between 0 and 1 as should for a probability.

Question 2: The infinite well

Consider an **infinite well** with potential $V(x) = 0$ for $x \in [0,1]$ and ∞ otherwise, and the time-dependent wave function $\Psi(x,t)$ defined with $x \in [0,1]$:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}(\psi_1(x)e^{-iE_1 t/\hbar} + \psi_2(x)e^{-iE_2 t/\hbar}), \qquad \text{with eigenstates } \psi_n(x) = \sqrt{2}\sin(n\pi x)$$

(a) Derive an expression for the probability density function $|\Psi(x,t)|^2$. Simplify as much as possible.
(b) Then calculate the time-dependent expectation value for the position $\langle x \rangle$.

Solution (Q2)

(a) First we derive the probability density function. To simplify the notation we denote $\omega_n \equiv E_n/\hbar = n^2 E_1/\hbar$ and $\Delta\omega = \omega_2 - \omega_1 = 4\omega_1 - \omega_1 = 3\omega_1$:

$$\begin{aligned} |\Psi(x,t)|^2 &= \frac{1}{2} \left| (\psi_1(x)e^{-iE_1 t/\hbar} + \psi_2(x)e^{-iE_2 t/\hbar}) \right|^2 \\ &= \frac{1}{2} \left(\psi_1^*(x)e^{i\omega_1 t} + \psi_2^*(x)e^{i4\omega_1 t} \right) \left(\psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i4\omega_1 t} \right) \\ &= \frac{1}{2} \left[|\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1^*(x)\psi_2(x)e^{-i\Delta\omega t} + +\psi_1(x)\psi_2^*(x)e^{i\Delta\omega t} \right] \\ &= \frac{1}{2} \left[|\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1(x)\psi_2(x) \left(e^{-i\Delta\omega t} + e^{i\Delta\omega t} \right) \right] \\ &= \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \psi_1(x)\psi_2(x)\cos(\Delta\omega\, t) \\ &= \sin^2(\pi x) + \sin^2(2\pi x) + 2\sin(\pi x)\sin(2\pi x)\cos(\Delta\omega\, t) \end{aligned}$$

We see that the probability density function is oscillating in time.

(b) To calculate the expectation value of the position, we can use the probability density function obtained in (a):

$$
\begin{aligned}
\langle x \rangle &= \int_0^1 x\, |\Psi(x,t)|^2\, dx \\
&= \frac{1}{2}\int_0^1 x\, |\psi_1(x)|^2\, dx + \frac{1}{2}\int_0^1 x\, |\psi_2(x)|^2\, dx + \cos(\Delta\omega\, t)\int_0^1 x\, \psi_1(x)\psi_2(x)\, dx \\
&= \int_0^1 x\, \sin^2(\pi x)\, dx + \int_0^1 x\, \sin^2(2\pi x)\, dx + 2\cos(\Delta\omega\, t)\int_0^1 x\, \sin(\pi x)\sin(2\pi x)\, dx \\
&= \frac{1}{4} + \frac{1}{4} - 2\cos(\Delta\omega\, t)\frac{8}{9\pi^2} \\
&= \frac{1}{2} - \frac{16}{9\pi^2}\cos(\Delta\omega\, t)
\end{aligned}
$$

The expectation value for the position is oscillating around the well center with angular frequency $\Delta\omega = (E_2 - E_1)/\hbar$.

Question 3: Ladder operators for the harmonic oscillator

Consider a **harmonic oscillator** in the following superposition state $\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ state.

(a) Derive an expression for $\hat{a}_-\psi$ as function of eigenstates ψ_n.
(b) Calculate the expectation value for the lowering operator $\langle \hat{a}_- \rangle$.

Hint: Eigenstates are orthonormal, and $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$.

Solution (Q3)

(a) Applying the lowering operator onto ψ leads to:

$$\hat{a}_-\psi = \frac{1}{\sqrt{2}}(\hat{a}_-\psi_1 + \hat{a}_-\psi_2) = \frac{1}{\sqrt{2}}\left(\psi_0 + \sqrt{2}\psi_1\right),$$

which is an expression in only the eigenstates, although not a normalized state.

(b) The expectation value of the lowering operator can be derived as follows:

$$
\begin{aligned}
\langle \hat{a}_- \rangle &= \langle \psi | \hat{a}_- | \psi \rangle \\
&= \frac{1}{2}\left(\langle \psi_1 | + \langle \psi_2 |\right)\, \hat{a}_-\, \left(|\psi_1\rangle + |\psi_2\rangle\right) \\
&= \frac{1}{2}\left(\langle \psi_1 | + \langle \psi_2 |\right)\left(|\psi_0\rangle + \sqrt{2}|\psi_1\rangle\right) \\
&= \frac{1}{2}\left(\sqrt{2}\langle \psi_1|\psi_1\rangle + \sqrt{2}\langle \psi_2|\psi_1\rangle + \langle \psi_1|\psi_0\rangle + \langle \psi_2|\psi_0\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle \psi_1|\psi_1\rangle = \frac{1}{\sqrt{2}}
\end{aligned}
$$

Remark that the lowering operator is not an Hermitian operator, hence the expectation value of the lowering operator is not necessarily measurable. The eigenstates are the coherent states and its eigenvalues can be complex. However, for this question we just calculate the expectation value and ignore any physical interpretation difficulties.

Question 4: Hydrogen atom

Assume a **hydrogen atom** in the ground state:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}, \qquad \text{with } a \text{ the Bohr radius.}$$

(a) What is the expectation value for the radius squared: $\langle r^2 \rangle$?
(b) What is the expectation value for the energy $\langle H \rangle$ if the system is in superposition state: $\psi = \frac{1}{3}(2\psi_{100} + \psi_{210} + 2\psi_{500})$? *Hint:* ψ_{nlm} are the hydrogen atom eigenstates and corresponding eigenenergies $E_n = -\text{Ry}/n^2 = -13.6/n^2$ eV.

Solution (Q4)

(a) The expectation value $\langle r^2 \rangle$ is given by:

$$\begin{aligned}
\langle r^2 \rangle &= \iiint_{\mathbb{R}^3} r^2 |\psi_{100}|^2\, dV \\
&= \int_0^{+\infty} r^4 |\psi_{100}(r,\theta,\phi)|^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \\
&= 4\pi\, \frac{1}{\pi a^3} \int_0^{+\infty} r^4 e^{-2r/a}\, dr \\
&= \frac{4}{a^3}\, \frac{4!\, a^5}{32} = 3a^2
\end{aligned}$$

(b) The expectation value $\langle H \rangle$ for $\psi = \frac{1}{3}(2\psi_{100} + \psi_{210} + 2\psi_{500})$ can be derived by expanding in the eigenenergies:

$$\begin{aligned}
\langle H \rangle &= |c_{100}|^2 E_{100} + |c_{210}|^2 E_{210} + |c_{500}|^2 E_{500} \\
&= \frac{4}{9} E_{100} + \frac{1}{9} E_{210} + \frac{4}{9} E_{500} \\
&= -\frac{4}{9}\text{Ry} - \frac{1}{9\cdot 4}\text{Ry} - \frac{4}{9\cdot 25}\text{Ry} \\
&= -\left(\frac{4}{9} + \frac{1}{9\cdot 4} + \frac{4}{9\cdot 25}\right)\text{Ry} \\
&= -\frac{441}{9\cdot 100}\,(400 + 25 + 16)\ \text{Ry} = -\frac{49}{100}\,\text{Ry} = -6.664\,\text{eV}
\end{aligned}$$

Question 5: Spin 1/2 particles

Consider a spin 1/2 particle in state $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix}$ (in the standard basis $\{|\uparrow\rangle, |\downarrow\rangle\}$).

(a) What is the probability of getting $\hbar/2$ when measuring the z-component of the spin of the particle: S_z?
(b) Calculate the expectation value $\langle\hat{S}_x\rangle = \langle\chi|\hat{S}_x|\chi\rangle$ where $\hat{S}_x$ is represented by the matrix $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$.

Solution (Q5)

(a) The eigenstates of $\hat{S}_z$ are the spin-up and spin-down states, with eigenvalues $\hbar/2$ and $-\hbar/2$. The operator is Hermitian and the probabilities are given by $|c_n|^2$ with c_n the coefficients of the eigenstate expansion:

$$|\chi\rangle \longrightarrow \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{2i}{\sqrt{5}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{5}}|\uparrow\rangle + \frac{2i}{\sqrt{5}}|\downarrow\rangle$$

Therefore, when measuring S_z the probability to observe:

- $\hbar/2$ is 1/5 and
- $-\hbar/2$ is 4/5.

(b) The expectation value of $\hat{S}_x$ is given by:

$$\langle\hat{S}_x\rangle = \langle\chi|\hat{S}_x|\chi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2i\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix} = \frac{\hbar}{10}\begin{pmatrix}1 & -2i\end{pmatrix}\begin{pmatrix}2i\\1\end{pmatrix} = \frac{\hbar}{10}(2i - 2i) = 0$$

Question 6: Two-level atom

Consider a particle in a two-level atom with Hamiltonian: $H = \begin{pmatrix}3 & 4i\\-4i & -3\end{pmatrix}$

(a) Find the eigenenergies of this system.
(b) Then calculate the normalized eigenstates.

Solution (Q6)

Remark that in this question dimensionless units are used to simplify the problem, but for real problems the Hamiltonian should have units of energy.

(a) The eigenenergies E_1 and E_2 can be found by solving the eigenvalue equation $H\psi = \lambda\psi$ with solutions ψ_n column vectors in $\mathbb{C}^2$.

$$(\lambda\mathbb{1} - H)\psi = 0 \quad \Rightarrow \quad \det(\lambda\mathbb{1} - H) = 0$$

We can factorize the determinant to obtain solutions for λ:

$$\begin{vmatrix} \lambda - 3 & -4i \\ 4i & \lambda + 3 \end{vmatrix} = (\lambda - 3)(\lambda + 3) - 16 = \lambda^2 - 25$$

and eigenergies are thus $E_1 = -5$ and $E_2 = 5$.

(b) For the eigenstates ψ_1 and ψ_2 we get:

$\boxed{E_1 = -5}$:

$$\begin{pmatrix} 3 & 4i \\ -4i & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad y = -2ix \quad \Rightarrow \quad \psi_1^{(0)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

where the normalization constant $A = 1/\sqrt{5}$ is found by setting :

$$1 = \langle \psi_1 | \psi_1 \rangle = |A|^2 \begin{pmatrix} 1 & 2i \end{pmatrix} \begin{pmatrix} 1 \\ -2i \end{pmatrix} = |A|^2(1 - 4i^2) = 5|A|^2$$

$\boxed{E_2 = 5}$:

$$\begin{pmatrix} 3 & 4i \\ -4i & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad x = 2iy \quad \Rightarrow \quad \psi_1^{(0)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

where again the normalization constant $A = 1/\sqrt{5}$ is found by setting :

$$1 = \langle \psi_2 | \psi_2 \rangle = |A|^2 \begin{pmatrix} -2i & 1 \end{pmatrix} \begin{pmatrix} 2i \\ 1 \end{pmatrix} = |A|^2(-4i^2 + 1) = 5|A|^2$$

Question 7: Bosons in an Harmonic Oscillator

Assume two noninteracting bosons in a 1D harmonic oscillator potential (ignore spin). One particle is in state ψ_0 and the other in ψ_1. Exchange forces will adjust the expectation value for the distance between particles:

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle_0 + \langle x^2 \rangle_1 - 2\langle x \rangle_0 \langle x \rangle_1 - 2|\langle x \rangle_{01}|^2 = \frac{\hbar}{2m\omega} + \frac{3\hbar}{2m\omega} + 0 - 2|\langle x \rangle_{01}|^2$$

where we filled in the expectation values $\langle x^2 \rangle_n = \langle \psi_n | x^2 | \psi_n \rangle = (n + 1/2)\frac{\hbar}{m\omega}$ and are only left with the unknown overlap integral: $\langle x \rangle_{01} = \int_{-\infty}^{+\infty} x \psi_0 \psi_1 dx$.

(a) Explain why the third term is equal to zero.
(b) Calculate the last term of the exchange forces: $2|\langle x \rangle_{01}|^2$. Check whether you have a meaningful value.

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} = \alpha\, e^{-\beta^2 x^2/2} \qquad \text{with } \alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}, \quad \beta = \sqrt{\frac{m\omega}{\hbar}}$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2} = \alpha\sqrt{2}\,\beta x\, e^{-\beta^2 x^2/2}$$

Solution (Q7)

(a) The third term $2\langle x\rangle_0\langle x\rangle_1$ is equal to zero because the expectation value of any eigenstate $\psi_n(x)$ is zero for the harmonic oscillator, which has a symmetric potential and eigenstates that are either even or odd around zero. We can show this also by explicit calculation of $\langle x\rangle_0$:

$$\langle x\rangle_0 = \int_{-\infty}^{+\infty} x|\psi_0|^2\, dx = \alpha^2 \int_{-\infty}^{+\infty} xe^{-\beta^2 x^2}\, dx = 0$$

Where the last integral is zero because the integrand is an odd function of x around zero.

(b) The last term of the expectation value $2|\langle x\rangle_{01}|^2$ can be calculated:

$$\begin{aligned}
\langle x\rangle_{01} &= \int_{-\infty}^{+\infty} x\psi_0\psi_1 dx \\
&= \int_{-\infty}^{+\infty} x\left(\alpha e^{-\beta^2x^2/2}\right)\left(\alpha\sqrt{2}\beta x e^{-\beta^2x^2/2}\right) dx \\
&= \alpha^2\beta\sqrt{2}\int_{-\infty}^{+\infty} x^2 e^{-\beta^2x^2} dx \\
&= \alpha^2\beta\sqrt{2}\,\frac{\sqrt{\pi}}{2(\beta^2)^{3/2}} \\
&= \frac{\alpha^2\sqrt{\pi}}{\sqrt{2}\beta^2} \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{\sqrt{\pi}}{\sqrt{2}}\,\frac{\hbar}{m\omega} = \sqrt{\frac{\hbar}{2m\omega}}
\end{aligned}$$

Therefore $2|\langle x\rangle_{01}|^2 = \frac{\hbar}{m\omega}$ and the total expected distance (squared) is positive and smaller than for distinguishable particles:

$$\langle \Delta x^2\rangle = \frac{2\hbar}{m\omega} - \frac{\hbar}{m\omega} = \frac{\hbar}{m\omega},$$

which is a positive value and has units of length squared as required.