PHOT 301 Fall (2025-2026) **Final exam** Date: 22 January 2026
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Student Name:

Grading: The final exam counts for 40% of your total grade.

Exam type: Closed-book, all questions can be answered **using only pen and paper**. Calculators, mobile phones, etc. are not allowed to be used during the exam.

The duration of the final exam is 3 hours.

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam. Please tell if any question is unclear or ambiguous.

Hints: For Hermitian $\hat{Q}$: $\hat{Q}\,\psi_n = q_n\,\psi_n \quad \Rightarrow \quad \exists c_n : \psi = \sum_n c_n \psi_n, \quad \langle Q \rangle = |c_n|^2 q_n$

For general operators: $\langle \hat{Q} \rangle = \langle \psi | \hat{Q} | \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(x)\, \hat{Q}\, \psi(x)\, dx$

Question 1: Wave functions

Consider the time-independent wave function $\psi(x)$ defined with $x \in \mathbb{R}$:

$$\psi(x) = Ax\, e^{-x}$$

(a) First calculate the normalization constant A of the wave function.
(b) Then calculate the probability to find the particle in interval $[-1, 1]$.

Question 2: The infinite well

Consider an **infinite well** with potential $V(x) = 0$ for $x \in [0, 1]$ and ∞ otherwise, and the time-dependent wave function $\Psi(x, t)$ defined with $x \in [0, 1]$:

$$\Psi(x, t) = \frac{1}{\sqrt{2}}(\psi_1(x) e^{-iE_1 t/\hbar} + \psi_2(x) e^{-iE_2 t/\hbar}), \qquad \text{with eigenstates } \psi_n(x) = \sqrt{2}\sin(n\pi x)$$

(a) Derive an expression for the probability density function $|\Psi(x, t)|^2$. Simplify as much as possible.
(b) Then calculate the time-dependent expectation value for the position $\langle x \rangle$.

Question 3: Ladder operators for the harmonic oscillator

Consider a **harmonic oscillator** in the following superposition state $\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ state.

(a) Derive an expression for $\hat{a}_-\psi$ as function of eigenstates ψ_n.
(b) Calculate the expectation value for the lowering operator $\langle \hat{a}_- \rangle$.

Hint: Eigenstates are orthonormal, and $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$.

Question 4: Hydrogen atom

Assume a **hydrogen atom** in the ground state:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}, \qquad \text{with } a \text{ the Bohr radius.}$$

(b) What is the expectation value for the radius squared: $\langle r^2 \rangle$?
(b) What is the expectation value for the energy $\langle H \rangle$ if the system is in superposition state: $\psi = \frac{1}{3}(2\psi_{100} + \psi_{210} + 2\psi_{500})$? *Hint:* ψ_{nlm} are the hydrogen atom eigenstates and corresponding eigenenergies $E_n = -\mathrm{Ry}/n^2 = -13.6/n^2$ eV.

Question 5: Spin 1/2 particles

Consider a spin 1/2 particle in state $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ (in the standard basis $\{|\uparrow\rangle, |\downarrow\rangle\}$).

(a) What is the probability of getting $\hbar/2$ when measuring the z-component of the spin of the particle: S_z?
(b) Calculate the expectation value $\langle \hat{S}_x \rangle = \langle \chi | \hat{S}_x | \chi \rangle$ where $\hat{S}_x$ is represented by the matrix $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Question 6: Two-level atom

Consider a particle in a two-level atom with Hamiltonian: $H = \begin{pmatrix} 3 & 4i \\ -4i & -3 \end{pmatrix}$

(a) Find the eigenenergies of this system.
(b) Then calculate the normalized eigenstates.

Question 7: Bosons in an Harmonic Oscillator

Assume two noninteracting bosons in a 1D harmonic oscillator potential (ignore spin). One particle is in state ψ_0 and the other in ψ_1. Exchange forces will adjust the expectation value for the distance between particles:

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle_0 + \langle x^2 \rangle_1 - 2\langle x \rangle_0 \langle x \rangle_1 - 2|\langle x \rangle_{01}|^2 = \frac{\hbar}{2m\omega} + \frac{3\hbar}{2m\omega} + 0 - 2|\langle x \rangle_{01}|^2$$

where we filled in the expectation values $\langle x^2 \rangle_n = \langle \psi_n | x^2 | \psi_n \rangle = (n + 1/2)\frac{\hbar}{m\omega}$ and are only left with the unknown overlap integral: $\langle x \rangle_{01} = \int_{-\infty}^{+\infty} x \psi_0 \psi_1 dx$.

(a) Explain why the third term is equal to zero.
(b) Calculate the last term of the exchange forces: $2|\langle x \rangle_{01}|^2$. Check whether you have a meaningful value.

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} = \alpha\, e^{-\beta^2 x^2/2} \qquad \text{with } \alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}, \quad \beta = \sqrt{\frac{m\omega}{\hbar}}$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2} = \alpha\sqrt{2}\,\beta x\, e^{-\beta^2 x^2/2}$$