PHOT 301: Quantum Photonics

Final exam questions (version C)

Michaël Barbier, Fall semester (2024-2025)

General information on the exam

Grading: This final exam will count for 50% of your total grade. Together with the projects which account for 40%, and the midterm exam that counts for 10%, your total grade for the course will be determined.

Exam type: The final exam consists of 8 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam. You are given an extra sheet with some formulas which can be used to help solve the problems.

The duration of the final exam is 3 hours.

Exam questions

Please answer all questions listed below. Each of the questions is valued equally in the score calculation of the exam. If a question contains multiple parts, each of the parts is valued equally within the score for the question.

Please tell if any question is unclear or ambiguous.

Question 1: Wave functions and expectation values

Consider the following 1D wave function defined for $x \in [0, \frac{1}{2}]$:

$$\psi(x) = A\, x\, \cos(\pi x)$$

with A a normalization constant.

(1/2) First calculate the normalization constant A of the wave function.
(2/2) Then calculate the expectation value $\langle \frac{1}{\hat{x}^2} \rangle$.

Question 2: Probability current in an infinite well

Consider the following superpositon of 1D infinite well eigenstates with time-dependent factors:

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \left(\psi_1(x)\, e^{-i\omega t} + \psi_2(x)\, e^{-i4\omega t} \right)$$

where $\omega = E_1/\hbar$ and the eigenstates and eigenenergies can be written as:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\, \sin\left(\frac{n\,\pi\,x}{L}\right), \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

(1/1) calculate the probability current density j which is given by the formula:

$$j(x,t) = -\frac{i\hbar}{2m} \left(\Psi^* \frac{d\Psi}{dx} - \Psi \frac{d\Psi^*}{dx} \right) = \frac{\hbar}{m}\, \Im \left\{ \Psi^* \frac{d\Psi}{dx} \right\}$$

where $\Im\{\dots\}$ represents the imaginary part. Take the width of the well $L = 1$ for simplicity.

Question 3: Time-evolution hydrogen 1S + 2S orbitals

Consider a hydrogen atom in a superposition of a 1S and 2S eigenstate:

$$\Psi(\vec{r},t) = \frac{1}{\sqrt{5}} \left(\psi_{100} e^{-iE_1 t/\hbar} + 2\, \psi_{200} e^{-iE_2 t/\hbar} \right).$$

where the 1S and 2S orbital eigenstates are given by:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}, \qquad E_1 = -\mathrm{Ry} \approx -13.6\,\mathrm{eV}$$

$$\psi_{200}(r,\theta,\phi) = \frac{1}{\sqrt{8\pi a_0^3}} \left(1 - \frac{r}{2a_0}\right) e^{-r/(2a_0)}, \qquad E_2 = -\frac{\mathrm{Ry}}{4}$$

(1/1) Calculate the expectation value for the distance from the center r and its evolution in time:

$$\langle r \rangle(t) = \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\; r^2 \sin\theta \;\; r\, |\Psi(\vec{r},t)|^2$$

Question 4: Perturbation: hydrogen ground state

Consider a hydrogen atom perturbed along the z-axis by a perturbation part: $\hat{H}_p = \beta\, z^2$ with β a small real-valued number:

$$\hat{H} = \hat{H}_0 + \hat{H}_p = \hat{H}_0 + \beta\, z^2$$

where the unperturbed Hamiltonian $\hat{H}_0$ has eigenenergies $E_n^{(0)}$, that is:

$$\hat{H}_0 \psi_{nlm}(x) = E_n^{(0)} \psi_{nlm}(x), \quad \text{and ground state:} \quad \psi_{100} = \frac{1}{\sqrt{\pi a^3}}\, e^{-\frac{r}{a}}, \quad E_1^{(0)} = -\text{Ry}$$

where a represents the Bohr radius, and the Rydberg energy unit Ry ≈ 13.6 eV.

(1/1) Calculate the value of the energy of the perturbed ground state E_1 within first order perturbation theory approximation.

Hint: The perturbed eigenenergies E_n up to first order in perturbation theory are given by:

$$E_n = E_n^{(0)} + \langle \psi_{nlm} | \hat{H}_p | \psi_{nlm} \rangle$$

Question 5: Operator

Consider the spin of a particle with the following orthonormal basis of eigenstates:

$$|u\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |d\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider further the operator $\hat{Q} = \hat{\sigma}_z + \hat{\sigma}_y$, that is:

$$\hat{Q} = \hat{\sigma}_z + \hat{\sigma}_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(1/2) What is the result of acting on the eigenstate $|u\rangle$ with the operator $\hat{Q}$

$$\hat{Q}|u\rangle = ?$$

(2/2) What happens if you act twice on the eigenstate $|u\rangle$ with the operator $\hat{Q}$, that is:

$$\hat{Q}^2|u\rangle = ?$$

Question 6: Spin in a magnetic field

A B-field is oriented along the diagonal between the x and y-axis: $\vec{B} = \frac{B}{\sqrt{2}}(1, 1, 0)$. That is, consider the following Schrodinger (Pauli) equation:

$$\mu_B \frac{B}{\sqrt{2}} \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} c_u \\ c_d \end{pmatrix} = E \begin{pmatrix} c_u \\ c_d \end{pmatrix}$$

where $\mu_B = \frac{e\hbar}{2m_0}$ is the Bohr magneton.

(1/1) Calculate the eigenenergies and corresponding eigenstates of the system.

Question 7: Periodic systems

Assume a 1D periodic system of equi-distant potential barriers. A single unit cell has length L and contains a single potential barrier. The relation between the coefficients (A, B) before the barrier at $x = 0$ and (C, D) after the barrier at $x = L$ can be written as:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

According to the Bloch theorem we can write

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} e^{i\beta L} & 0 \\ 0 & e^{i\beta L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

with β the Bloch wave vector which provides the slowly varying phase of the envelope function. Combining the above two conditions leads to a single matrix equation.

(1/1): Derive the characteristic equation (which defines the band structure $E(\beta)$), by combining the above equations and solving. You can use the fact that the determinant of the transfer-matrix equals one: $\det(T) = 1$. Bring the equation in the form of:

$$\cos(\beta L) = ...$$

where the right-hand-side of the equation does not depend on β.

Question 8: Two bosons in a harmonic oscillator

Consider a 1D harmonic oscillator with two identical bosonic particles. For two bosons the wave function is the symmetric sum of the single-particle functions of the two states the particles are in, a and b:

$$\psi(x_1, x_2) = A\,(\psi_a(x_1)\psi_b(x_2) + \psi_a(x_2)\psi_b(x_1)),$$

where x_1 is the x-coordinate of particle 1 and x_2 is the x-coordinate of particle 2.

(1/2) Derive the normalization constant A in case both particles are in the ground state: $\psi_a(x) = \psi_b(x) = \psi_0(x)$

(2/2) Derive the normalization constant A in case one particle is in the ground state and the other is in the first excited state: $\psi_a(x) = \psi_0(x), \quad \psi_b(x) = \psi_1(x)$

Hint: The condition for a normalized two particle wave function $\psi(x_1, x_2)$ is:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x_1, x_2)|^2\, dx_1\, dx_2 = 1$$

Further, you can make use of the fact that the single-particle eigenstates $\psi_n(x)$ are orthonormal and real-valued.