PHOT 301: Quantum Photonics

Final exam questions (version B)

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General information on the exam

Grading: This final exam will count for 50% of your total grade. Together with the projects which count for 40%, and the midterm exam that counts for 10%, your total grade for the course will be determined.

Exam type: The final exam consists of 8 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam. The last page of the exam contains some formulas which can be used to help solve the problems.

The duration of the final exam is 3 hours.

Exam questions

Please answer all questions listed below. Each of the questions is valued equally in the score calculation of the exam. If a question contains multiple parts, each of the parts is valued equally within the score for the question.

Please tell if any question is unclear or ambiguous.

Question 1: Wave functions and expectation values

Consider the following 1D wave function defined with $x \in [0, 1]$:

$$\psi(x) = A\,(x-1)\,\sin(3\pi x)$$

with A a normalization constant.

(1/2) First calculate the normalization constant A of the wave function.
(2/2) Then show that the expectation value of the momentum $\langle \hat{p} \rangle = 0$ with $\hat{p} = -i\hbar \frac{d}{dx}$.

Question 2: Probability current in Harmonic oscillator

Consider the following superpositon of 1D harmonic oscillator eigenstates with time-dependent factors:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_0\, e^{-i\omega t/2} + \psi_1\, e^{-i3\omega t/2}\right)$$

where the eigenstates can be written as:

$$\psi_0 = \alpha\, e^{-\beta^2 x^2/2}, \qquad \psi_1 = \sqrt{2}\alpha\beta\, x\, e^{-\beta^2 x^2/2}, \quad \text{with} \quad \alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}, \quad \beta = \sqrt{\frac{m\omega}{\hbar}}$$

(1/1) calculate the probability current density j which is given by the formula:

$$j = -\frac{i\hbar}{2m}\left(\Psi^* \frac{d\Psi}{dx} - \Psi\frac{d\Psi^*}{dx}\right) = \frac{\hbar}{m}\,\Im\left\{\Psi^*\frac{d\Psi}{dx}\right\}$$

Question 3: Time-evolution hydrogen 1S + 2S orbitals

Consider a hydrogen atom in a superposition of a 1S and 2S eigenstate:

$$\Psi(\vec{r},t) = \frac{1}{\sqrt{2}}\left(\psi_{100}e^{-iE_1 t/\hbar} + \psi_{200}e^{-iE_2 t/\hbar}\right).$$

where the 1S and 2S orbital eigenstates are given by:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}, \qquad E_1 = -\text{Ry} \approx -13.6\,\text{eV}$$

$$\psi_{200}(r,\theta,\phi) = \frac{1}{\sqrt{8\pi a_0^3}}\left(1 - \frac{r}{2a_0}\right) e^{-r/(2a_0)}, \qquad E_2 = -\frac{\text{Ry}}{4}$$

(1/1) Calculate the expectation value for the distance from the center r and its evolution in time:

$$\langle r\rangle(t) = \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, r^2 \sin\theta \;\; r\,|\Psi(\vec{r},t)|^2$$

Question 4: Perturbation of a three-state system

Consider a three-state system with following perturbed Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon \\ 0 & -\varepsilon & 0 \end{pmatrix}$$

with $\varepsilon \ll 1$ a positive real number.

(1/3) Extract the eigenenergies and eigenstates for the unperturbed Hamiltonian $\hat{H}_0$.
(2/3) Prove that the energy does not change within first order perturbation theory approximation. That is, prove that $\langle \psi_m^{(0)} | \hat{H}_p | \psi_m^{(0)} \rangle = 0$ for all eigenstates.
(3/3) Calculate the perturbed eigenstates $|\psi_m\rangle$ within first order perturbation approximation.

Hint: The perturbed eigenenergies E_m and eigenstates ψ_m up to first order in perturbation theory are given by:

$$E_m = E_m^{(0)} + \langle \psi_m^{(0)} | \hat{H}_p | \psi_m^{(0)} \rangle$$

$$|\psi_m\rangle = |\psi_m^{(0)}\rangle + \sum_{n \neq m} \frac{\langle \psi_n^{(0)} | \hat{H}_p | \psi_m^{(0)} \rangle}{E_m^{(0)} - E_n^{(0)}} |\psi_n^{(0)}\rangle$$

Question 5: Operator

Consider a particle with spin with the following orthonormal basis of eigenstates:

$$|u\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |d\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider further the operator $\hat{Q} = \hat{\sigma}_x - \hat{\sigma}_y$, that is:

$$\hat{Q} = \hat{\sigma}_x - \hat{\sigma}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(1/2) What is the result of acting on the eigenstate $|u\rangle$ with the operator $\hat{Q}$

$$\hat{Q}|u\rangle = ?$$

(2/2) What happens if you act twice on the eigenstate $|u\rangle$ with the operator $\hat{Q}$, that is:

$$\hat{Q}^2|u\rangle = ?$$

Question 6: Spin in a magnetic field

A B-field is oriented along the diagonal between the y and z-axis: $\vec{B} = \frac{B}{\sqrt{2}}(0, 1, 1)$. That is, consider the following Schrodinger equation:

$$\mu_B \frac{B}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_u \\ c_d \end{pmatrix} = E \begin{pmatrix} c_u \\ c_d \end{pmatrix}$$

where $\mu_B = \frac{e\hbar}{2m_0}$ is the Bohr magneton.

(1/2) Calculate the eigenenergies and corresponding eigenstates of the system.
(2/2) Do you get the same eigenenergies if you would rotate the magnetic field along the z-axis?

Question 7: Periodic systems

Assume a 1D periodic system of equi-distant potential barriers. A single unit cell has length L and contains a single potential barrier. The relation between the coefficients (A, B) before at $x = 0$ and (C, D) after the barrier at $x = L$ can be written as:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

According to the Bloch theorem we can write

$$\begin{pmatrix} C \\ D \end{pmatrix} = e^{i\beta L} \begin{pmatrix} A \\ B \end{pmatrix}$$

with β the Bloch wave vector which provides the slowly varying phase of the envelope function.

(1/1): Derive the characteristic equation (which defines the band structure $E(\beta)$) using the Bloch theorem. Split into a real and imaginary part of the equation by making use of the fact that the determinant of the transfer-matrix equals one: $\det(T) = 1$.

Question 8: Two fermions in a harmonic oscillator

Consider a 1D harmonic oscillator with two fermionic particles in the same spin state. Assume that the particles are not interacting and ignore exchange energy. In this case the decoupled Hamiltonian can be written as the sum of the single-particle Hamiltonians:

$$\hat{H} = \hat{H}_1 + \hat{H}_2 = \frac{\hat{p}_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2$$

Consider further that one of the single-particle functions is in the ground-state and one is in the first excited state. For two fermions the (normalized) wave function is the anti-symmetric sum (Slater determinant) of the single-particle functions:

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1))$$

(1/1) Prove that the energy of the system is the sum of the single-particle energies: $E = \frac{1}{2}\hbar\omega + \frac{3}{2}\hbar\omega = 2\hbar\omega$ by calculating the expectation value of the Hamiltonian:

$$\langle \hat{H} \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x_1, x_2)\hat{H}\psi(x_1, x_2)\, dx_1\, dx_2$$

Hint: For an harmonic oscillator the single-particle eigenenergies are given by:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad \text{with } n = 0, 1, 2, 3, ...$$

And you can make use of the fact that the single-particle eigenstates $\psi_n(x)$ are orthonormal and real-valued.

Formulas

In the following formulas parameters n, m are integers and $0 < a \in \mathbb{R}$ and $b \in \mathbb{R}_0$:

Anti-derivatives (indefinite integrals)

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\arctan(x) + \frac{x}{x^2+1}\right)$$
$$\int \frac{x}{(x^2+1)^2}\,dx = -\frac{1}{2}\frac{1}{x^2+1}$$
$$\int \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\arctan(x) - \frac{x}{x^2+1}\right)$$
$$\int \frac{x^3}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\frac{1}{x^2+1} + \log(x^2+1)\right)$$

$$\int \cos^n(ax)\sin(ax)\,dx = -\frac{1}{a(n+1)}\cos^{n+1}(ax)$$
$$\int \cos(ax)\sin^n(ax)\,dx = \frac{1}{a(n+1)}\sin^{n+1}(ax)$$

Definite integrals

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \frac{1}{2}\delta_{mn}$$
$$\int_0^1 x\sin^2(m\pi x)\,dx = \frac{1}{4}$$
$$\int_0^1 x^2\sin^2(m\pi x)\,dx = \frac{1}{6} - \frac{1}{4\pi^2 m^2}$$
$$\int_0^1 x\sin(\pi x)\sin(3\pi x)\,dx = 0$$
$$\int_0^1 x^2\sin(\pi x)\sin(3\pi x)\,dx = \frac{3}{16\pi^2}$$
$$\int_0^1 x^3\sin(\pi x)\sin(3\pi x)\,dx = \frac{9}{32\pi^2}$$

Definite integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$
$$\int_0^\infty e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$
$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$
$$\int_0^\infty x^4 e^{-ax^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$

$$\int_{-\infty}^\infty e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{\sqrt{a}}$$
$$\int_{-\infty}^\infty x^2 e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$
$$\int_{-\infty}^\infty x^4 e^{-ax^2}\,dx = \frac{3\sqrt{\pi}}{4a^{5/2}}$$

Integration in spherical coordinates:

$$\int_{-\infty}^\infty dx\int_{-\infty}^\infty dy\int_{-\infty}^\infty dz\, f(x,y,z) = \int_0^\infty d\rho\int_0^\pi d\theta\int_0^{2\pi} d\phi\,\rho^2\sin\theta\; F(\rho,\theta,\phi)$$

where volume element $dx\,dy\,dz = \rho^2\sin\theta\,d\theta\,d\phi\,d\rho$

$x = \rho\sin(\theta)\cos(\phi),\;\; y = \rho\sin(\theta)\sin(\phi),\;\; z = \rho\cos(\theta)$