PHOT 222: Quantum Photonics
LECTURE 12

Michaël Barbier, Spring semester (2024-2025)

OVERVIEW OF THE COURSE

week	topic	Serway 9th	Young
Week 1	Relativity	Ch. 39	Ch. 37
Week 2	Waves and Particles	Ch. 40	Ch. 38-39
Week 3	Wave packets and Uncertainty	Ch. 40	Ch. 38-39
Week 4	The Schrödinger equation and Probability	Ch. 41	Ch. 39
Week 5	**Midterm exam 1**		
Week 6	Quantum particles in a potential	Ch. 41	Ch. 40
Week 7	Bayram		
Week 8	Harmonic oscillator	Ch. 41	Ch. 40
Week 9	Tunneling through a potential barrier	Ch. 41	Ch. 40
Week 10	**Midterm exam 2**		
Week 11	Bohr's hydrogen atom, absorption/emission spectra	Ch. 42	Ch. 41
Week 12	Quantum mechanical model of the hydrogen atom	Ch. 42	Ch. 41
Week 13	Spin / Many-electron atoms / Atom bonds & molecules	Ch. 42-43	Ch. 41-42
Week 14	Crystalline materials and energy band structure	Ch. 44	Ch. 43

3D Particle in a Box

3D SCHRODINGER EQUATION

- Remember: time-independent Schrodinger equation in 1D:

$$\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial \mathrm{x}^2}} + U(x)\,\psi = E\,\psi$$

- First term corresponds to the kinetic energy K of the particle
- Kinetic energy K in 3D:

$$K = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \qquad \rightarrow \qquad \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_z^2}{2m}$$

- 3D time-independent Schrodinger equation:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial \mathrm{x}^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + U(x,y,z)\,\psi = E\,\psi$$

3D PARTICLE IN A BOX

- Schrodinger equation:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial \mathrm{x}^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}\right)+U(x,y,z)\,\psi = E\,\psi$$

- Separation of the variables to find $\psi(x,y,z)$:

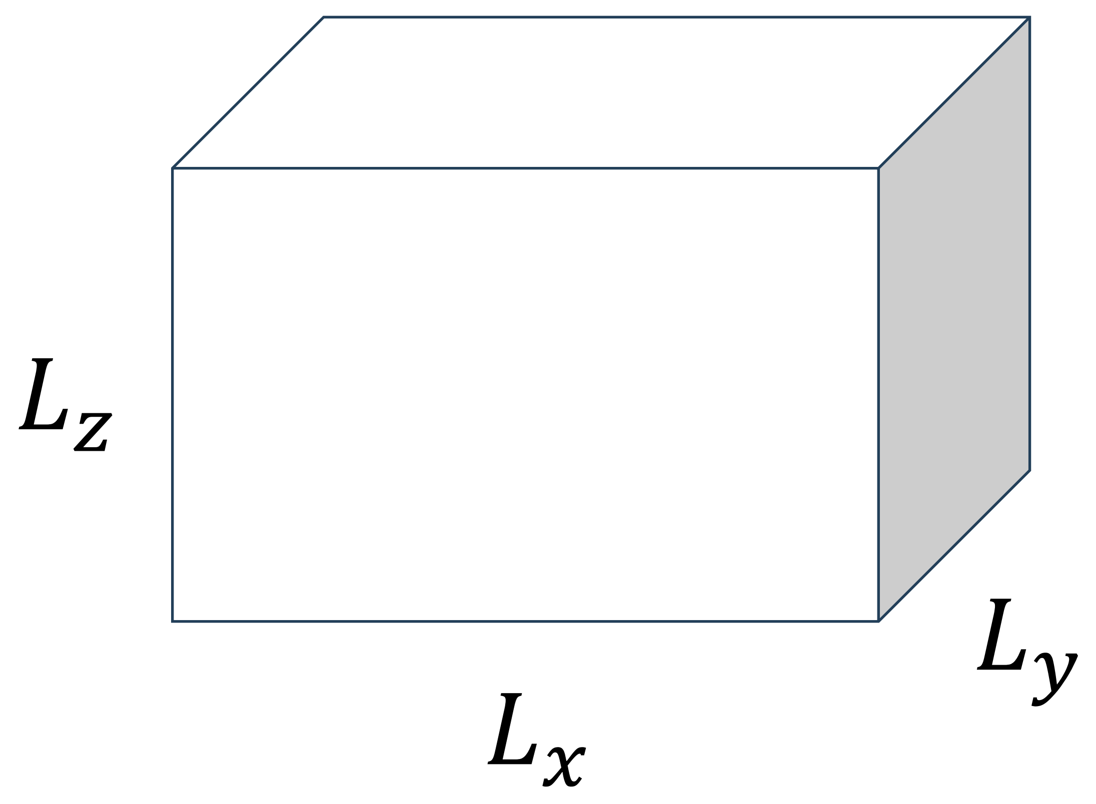
$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(\frac{\partial^2 X\,Y\,Z}{\partial \mathrm{x}^2}+\frac{\partial^2 X\,Y\,Z}{\partial y^2}+\frac{\partial^2 X\,Y\,Z}{\partial z^2}\right) = E\,\psi$$

Inside the box:

$$U(x,y,z) = 0$$

$$\begin{cases} 0 < x < L_x \\ 0 < y < L_y \\ 0 < z < L_z \end{cases}$$



3D PARTICLE IN A BOX: SEPARATION OF THE VARIABLES

- Separation of the variables to find $\psi(x, y, z)$:

$$\psi(x, y, z) = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(\frac{\partial^2 XYZ}{\partial \mathrm{x}^2} + \frac{\partial^2 XYZ}{\partial y^2} + \frac{\partial^2 XYZ}{\partial z^2}\right) = E\,\psi$$

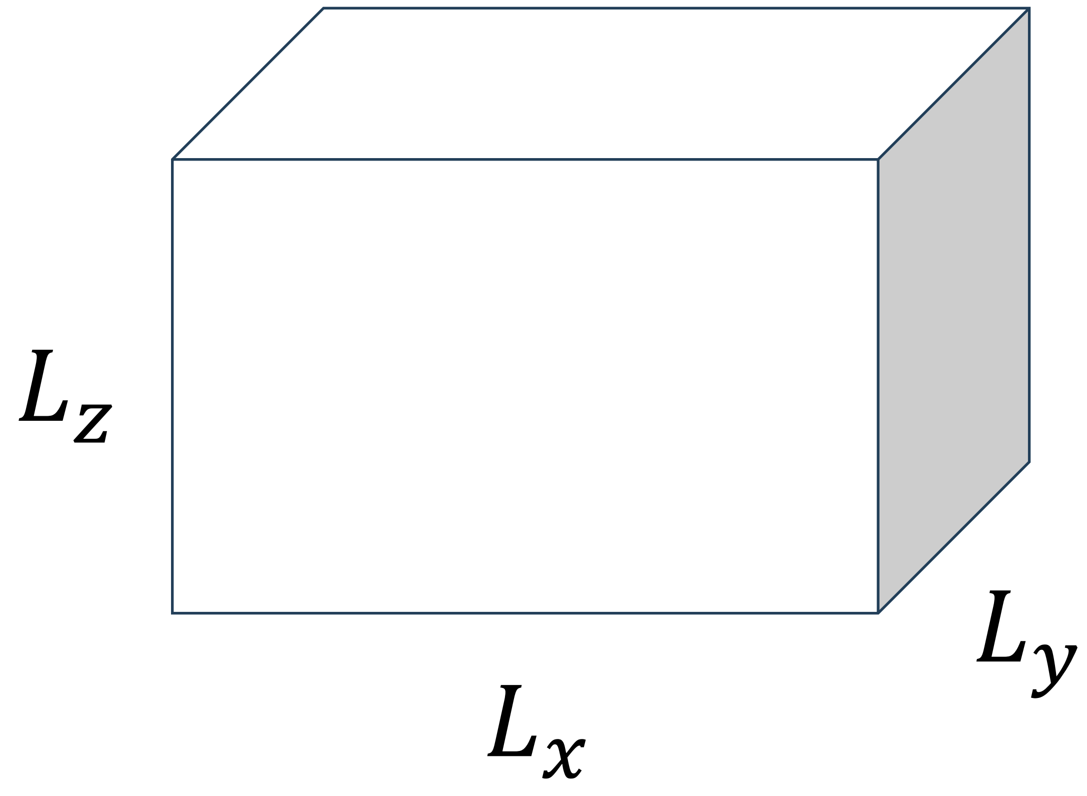
$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(YZ\frac{\partial^2 X}{\partial \mathrm{x}^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2}\right) = E\,\psi$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(\frac{1}{X}\frac{\partial^2 X}{\partial \mathrm{x}^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right) = E$$

Inside the box:

$$U(x, y, z) = 0$$

$$\begin{cases} 0 < x < L_x \\ 0 < y < L_y \\ 0 < z < L_z \end{cases}$$



3D PARTICLE IN A BOX: SEPARATION OF THE VARIABLES

$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

Inside the box:

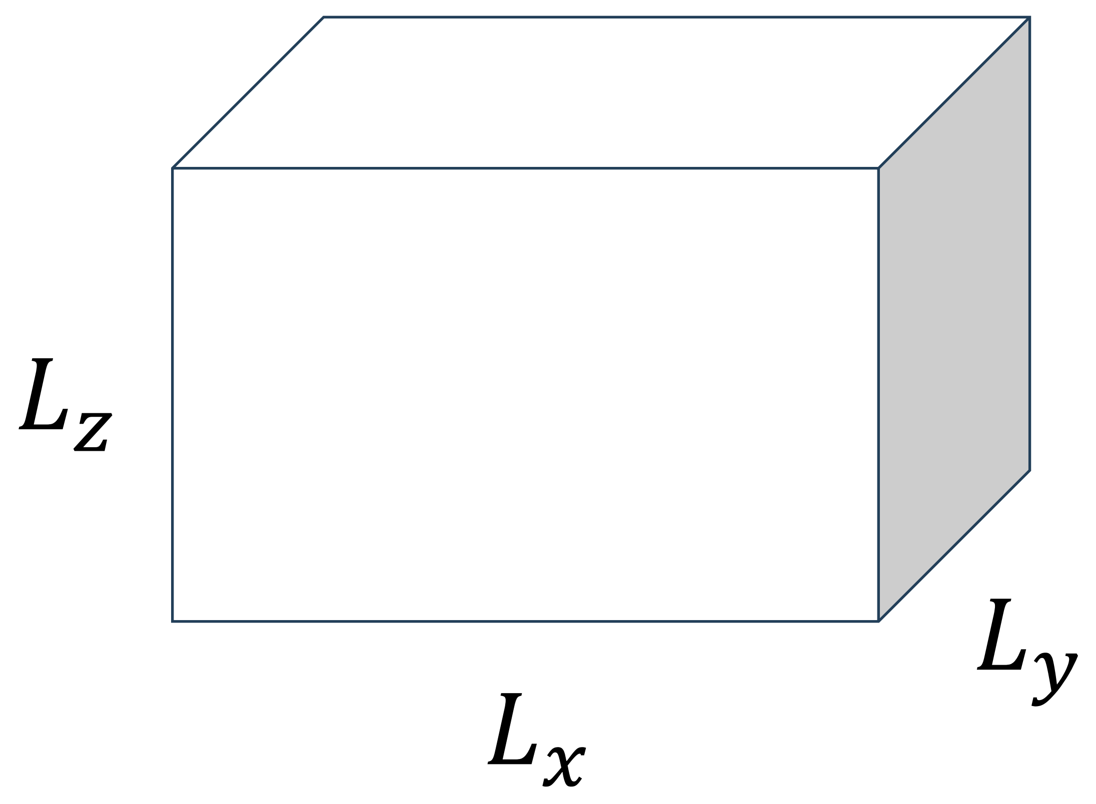
$$U(x,y,z) = 0$$

$$\begin{cases} 0 < x < L_x \\ 0 < y < L_y \\ 0 < z < L_z \end{cases}$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right) = E$$

System of equations:

$$\Rightarrow \quad \begin{cases} -\dfrac{\hbar^2}{2m}\dfrac{d^2X}{dx^2} = E_x\,X \\ -\dfrac{\hbar^2}{2m}\dfrac{d^2Y}{dy^2} = E_y\,Y \\ -\dfrac{\hbar^2}{2m}\dfrac{d^2Z}{dz^2} = E_z\,Z \end{cases} \quad , \quad E = E_x + E_y + E_z$$



3D PARTICLE IN A BOX: SOLUTIONS

- Each equation is similar to the 1D particle in a box

$$-\frac{\hbar^2}{2m}\frac{\partial^2 X}{\partial x^2} = E_x\,X, \qquad X(x) = \sqrt{\frac{2}{\mathrm{L_x}}}\,\sin\left(\frac{n_x \pi x}{L_x}\right), \qquad E_x = \frac{\hbar^2\pi^2 n_x^2}{2mL_x^2}$$

- Total wave function the **product**:

$$\psi(x,y,z) = X\;Y\;Z = \sqrt{\frac{8}{\mathrm{L_x L_y L_z}}}\,\sin\left(\frac{n_x \pi x}{L_x}\right)\,\sin\left(\frac{n_y \pi y}{L_y}\right)\,\sin\left(\frac{n_z \pi z}{L_z}\right)$$

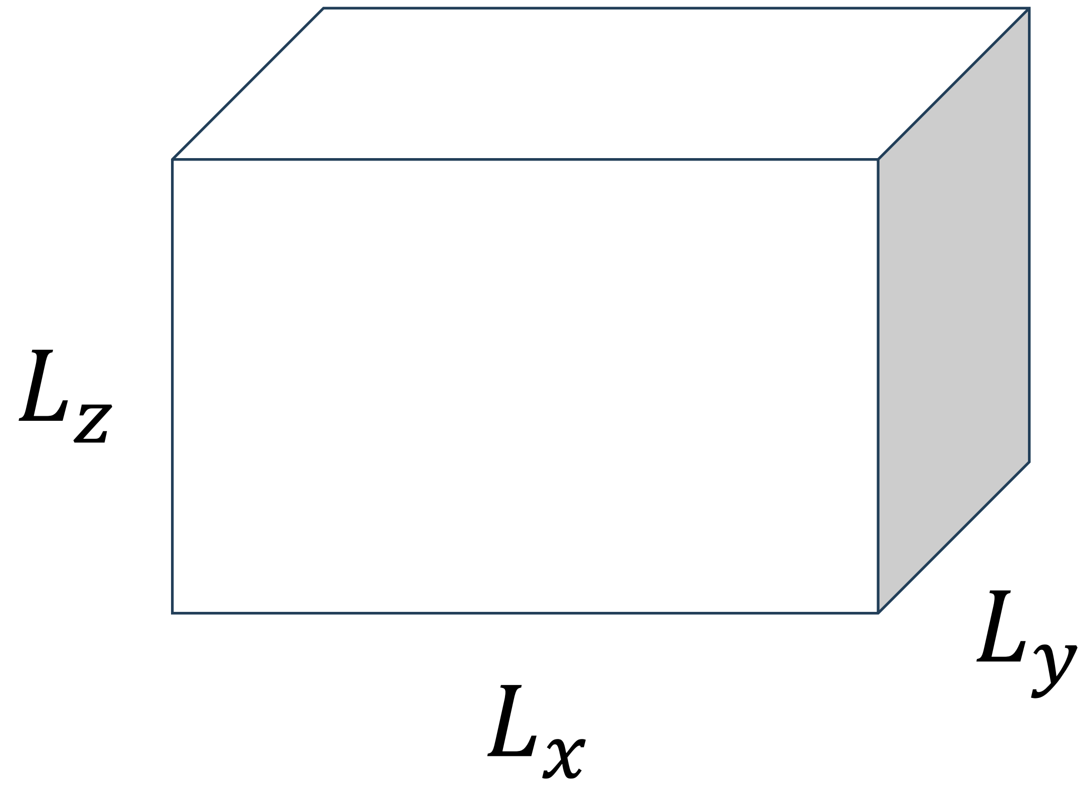
- With energy the **sum**:

$$E = E_x + E_y + E_z = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$$

Inside the box:

$$U(x,y,z) = 0$$

$$\begin{cases} 0 < x < L_x \\ 0 < y < L_y \\ 0 < z < L_z \end{cases}$$



3D PARTICLE IN A BOX: NORMALIZATION

- Normalization constant in 3D?

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x,y,z)|^2 \, dz\, dy\, dz$$

$$\Rightarrow \quad 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |X|^2\, |Y|^2\, |Z|^2 dz\, dy\, dz$$

$$\Rightarrow \quad 1 = \int_{-\infty}^{\infty} |X|^2 dx \times \int_{-\infty}^{\infty} |Y|^2 dy \times \int_{-\infty}^{\infty} |Z|^2 dz$$

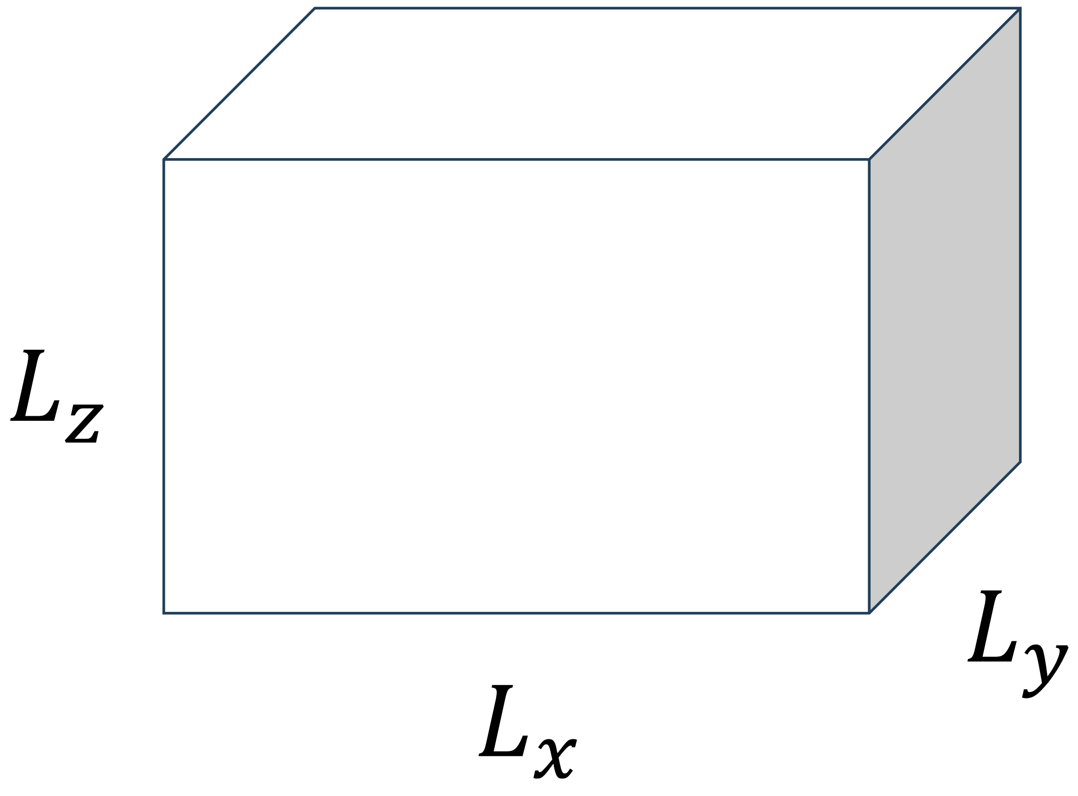
- Separation of variables: if all parts normalized then total wave function normalized

$$\psi(x,y,z) = X\,Y\,Z = \sqrt{\frac{8}{\mathrm{L_x L_y L_z}}}\,\sin\left(\frac{n_x \pi x}{L_x}\right)\sin\left(\frac{n_y \pi y}{L_y}\right)\sin\left(\frac{n_z \pi z}{L_z}\right)$$

Inside the box:

$$U(x,y,z) = 0$$

$$\begin{cases} 0 < x < L_x \\ 0 < y < L_y \\ 0 < z < L_z \end{cases}$$



3D PARTICLE IN A BOX: QUANTUM NUMBERS

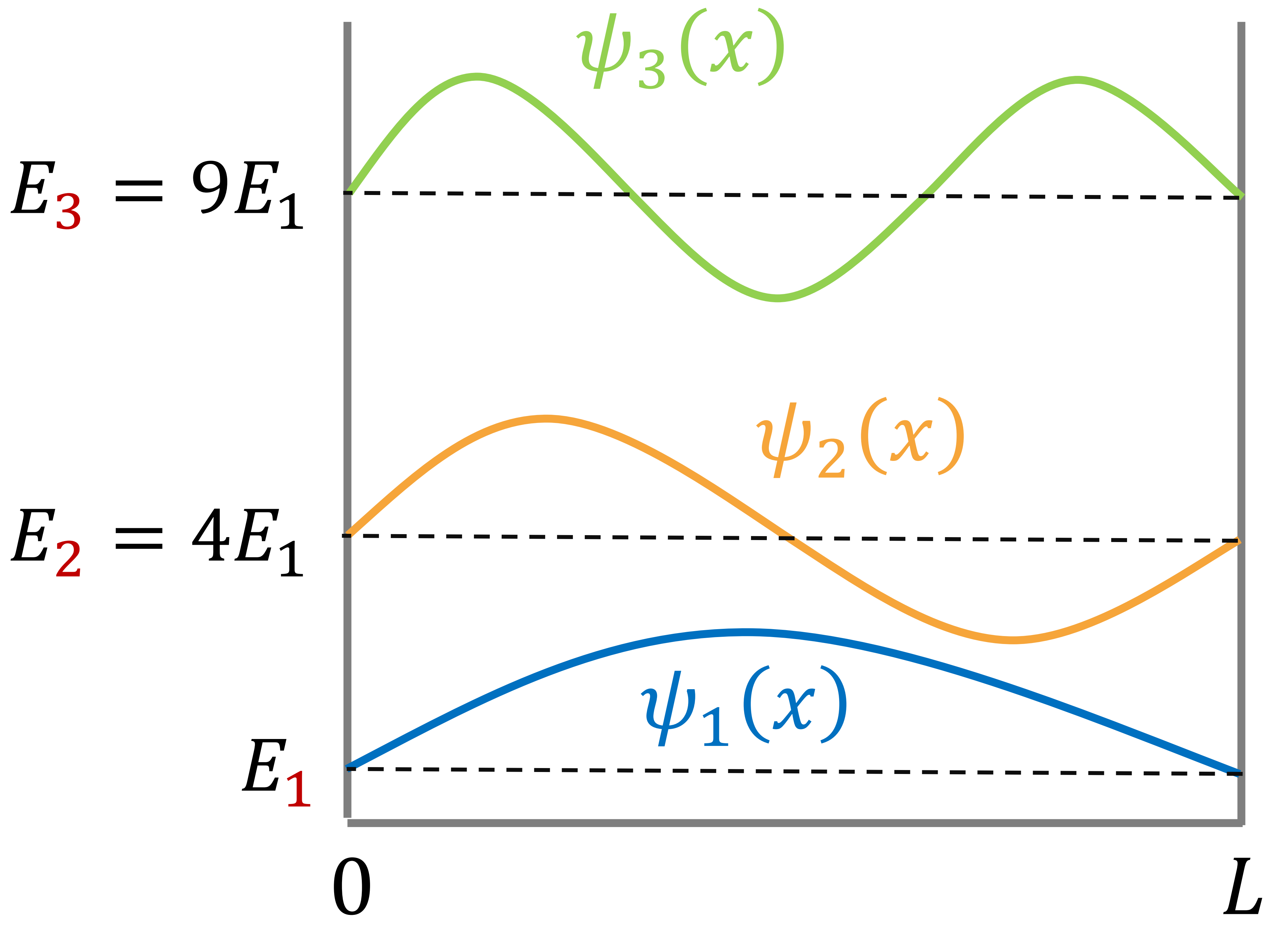
- In 1D we had quantum number n :

- Solutions for the wave function:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

- Corresponding energies:

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$



3D PARTICLE IN A BOX: QUANTUM NUMBERS

- In 1D we had quantum number n :

Particles "position" depending on n: $\psi_n(x)$, E_n

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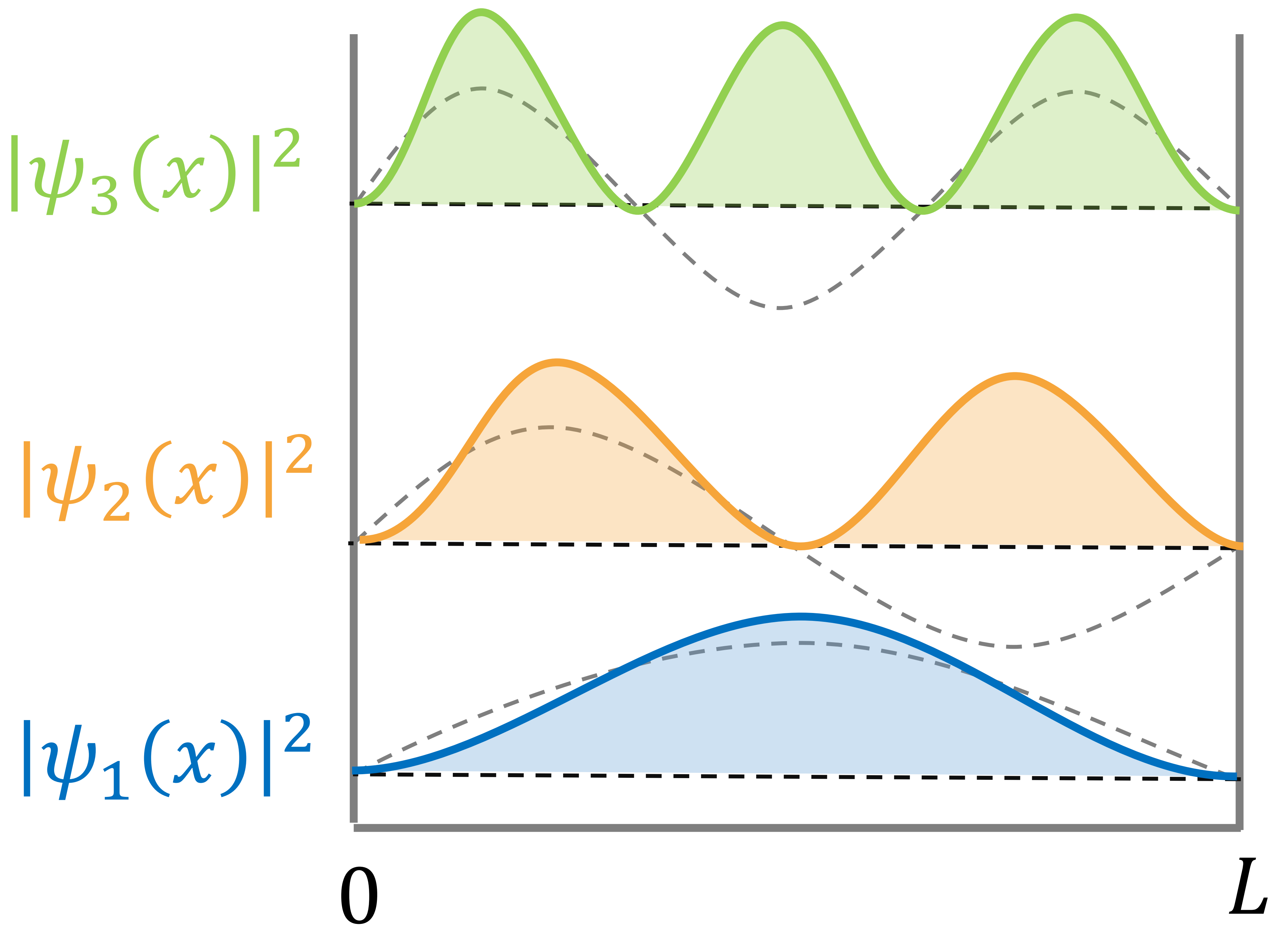


3D PARTICLE IN A BOX: STATIONARY SOLUTIONS

- Stationary solutions : $\psi_n(x)e^{\frac{iE_n t}{\hbar}}$

- Probability density does not change in time

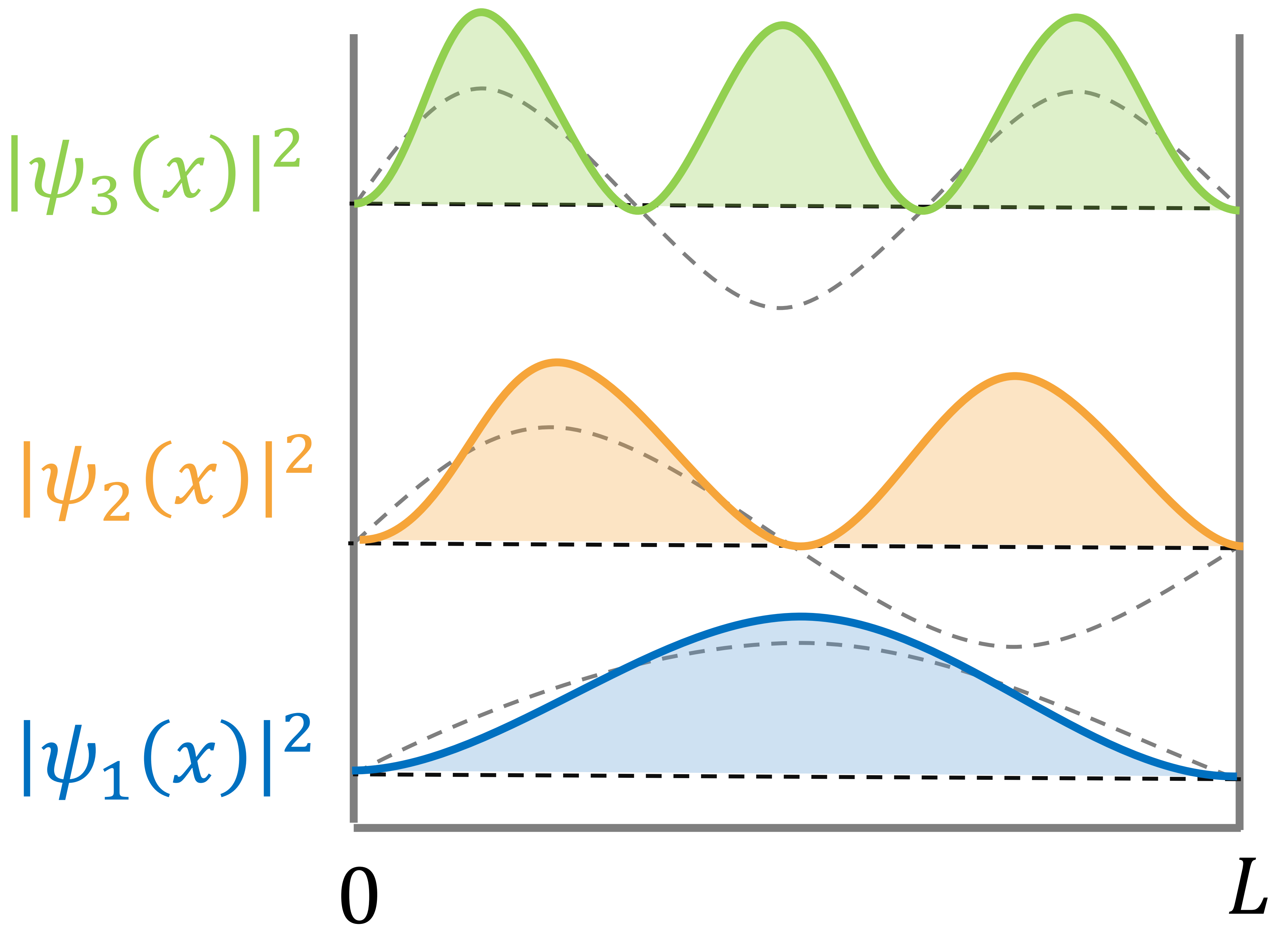
$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right),$$

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

- But superpositions not stationary:

$$\psi(x) = \psi_1(x)e^{\frac{iE_1 t}{\hbar}} + \psi_2(x)e^{\frac{iE_2 t}{\hbar}}$$

Particles "position" depending on n: $\psi_n(x)$, E_n



3D PARTICLE IN A BOX: STATIONARY SOLUTIONS

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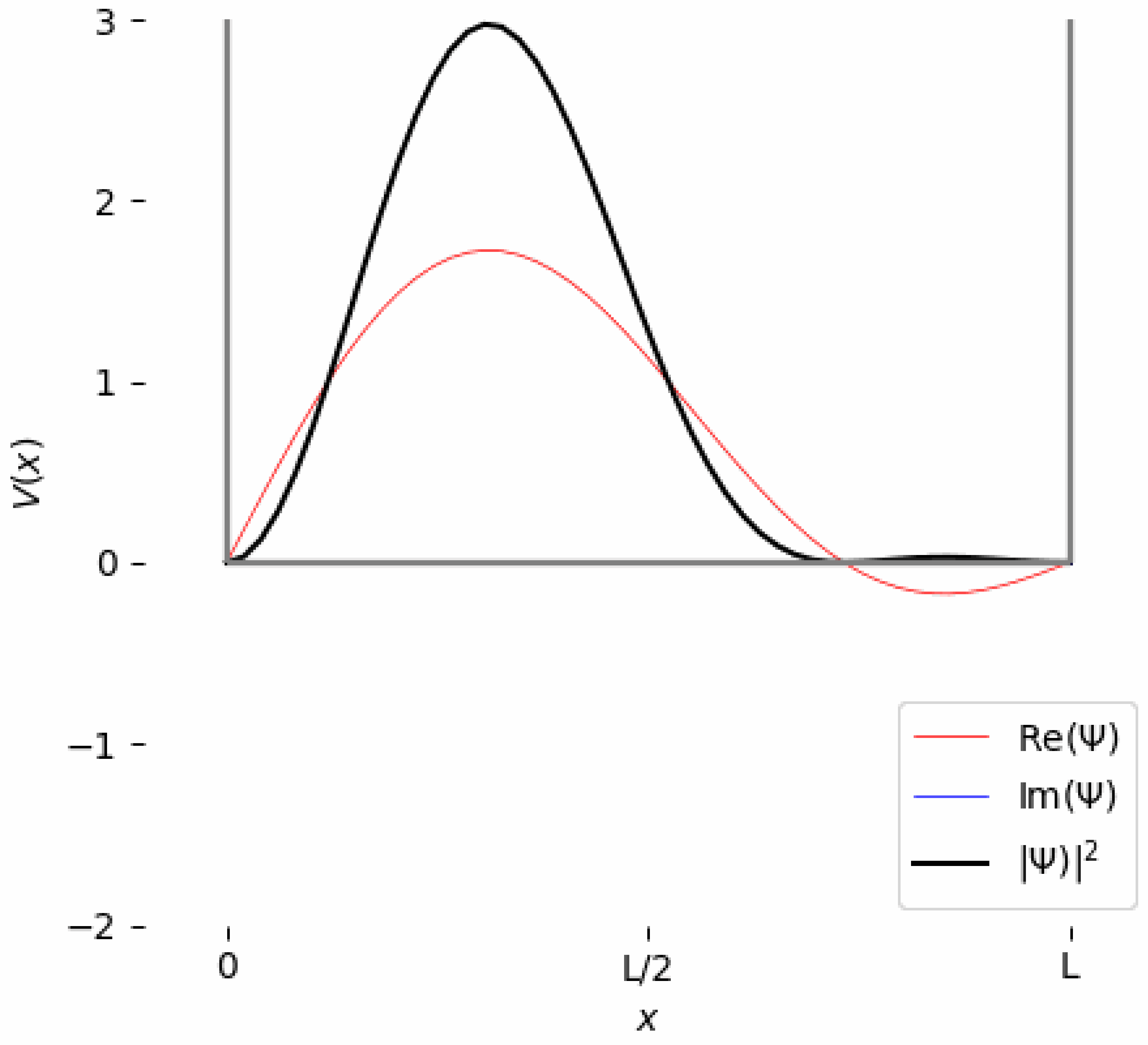


3D PARTICLE IN A BOX: QUANTUM NUMBERS

- In 1D we had quantum number n :

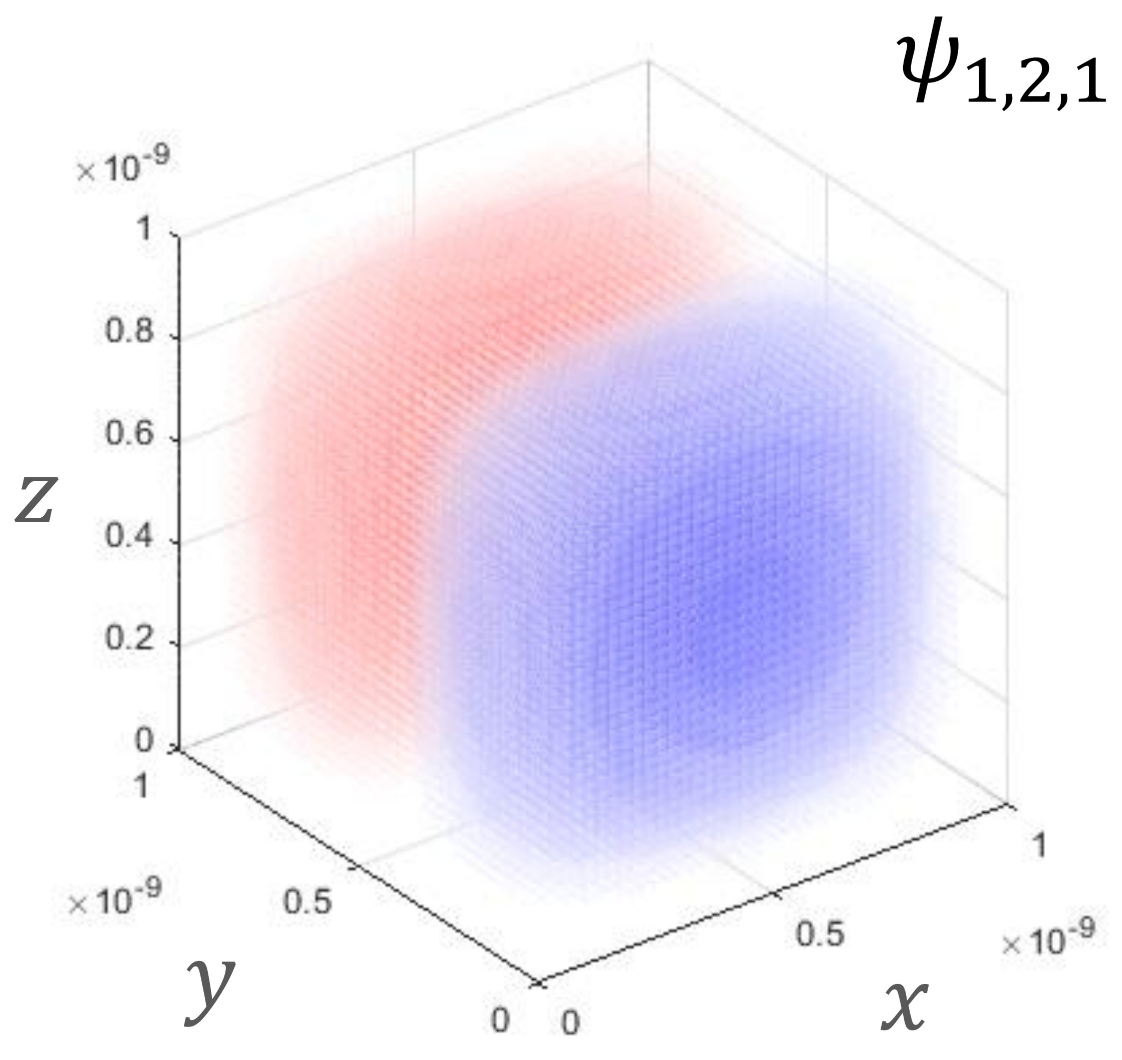
$$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

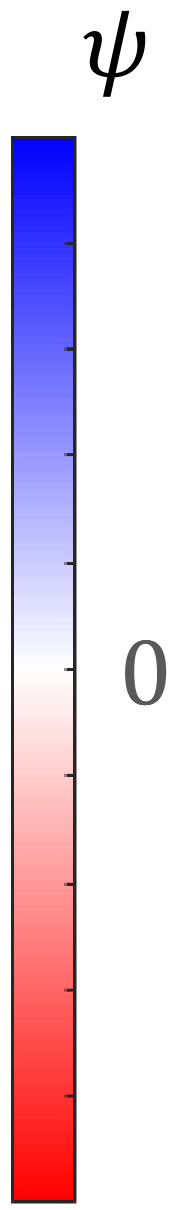
- In 3D we have **quantum numbers**: n_x, n_y, n_z :

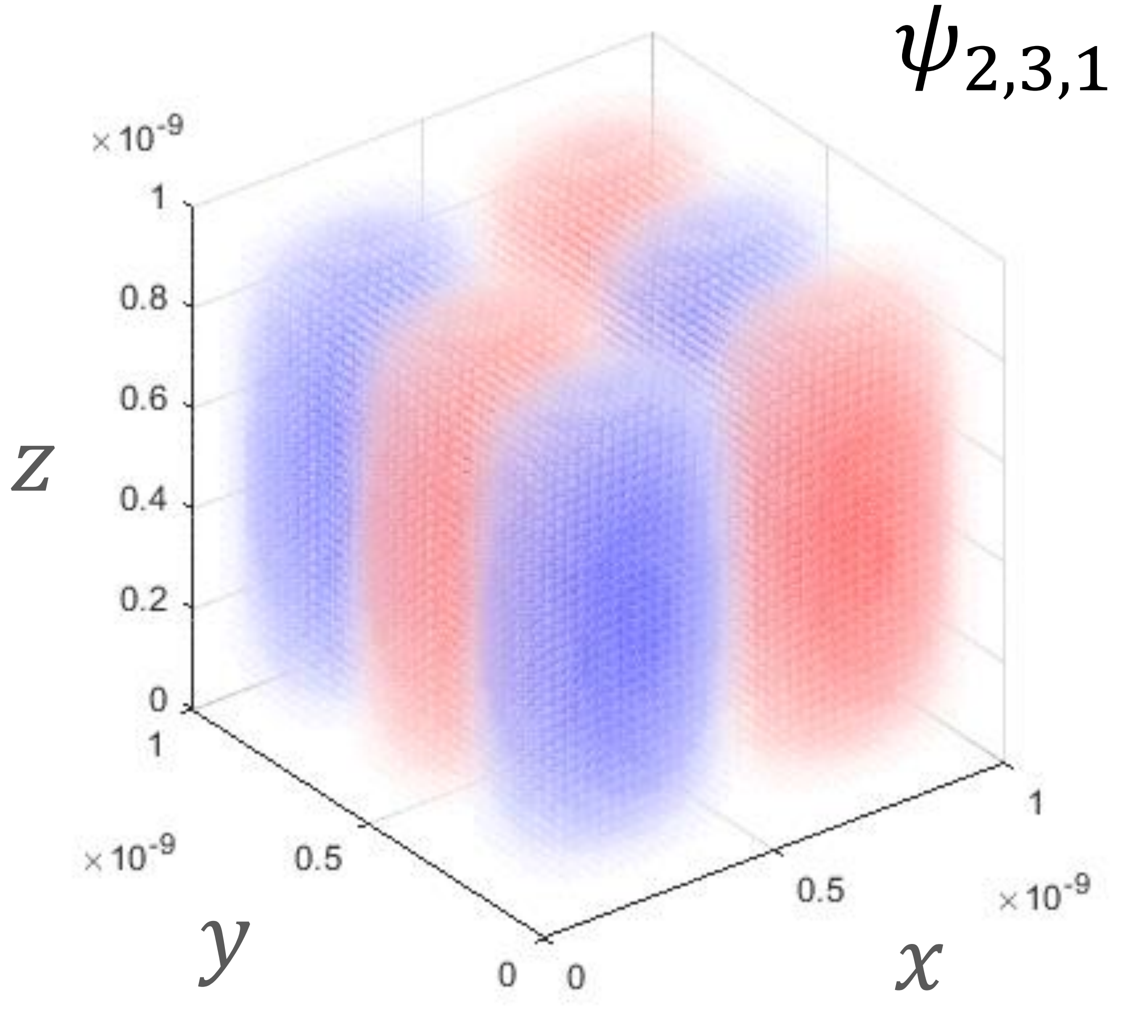
$$\psi_{n_x,n_y,n_z}(x,y,z) = \sqrt{\frac{8}{\mathrm{L_x L_y L_z}}}\,\sin\left(\frac{n_x\pi x}{L_x}\right)\,\sin\left(\frac{n_y\pi y}{L_y}\right)\,\sin\left(\frac{n_z\pi z}{L_z}\right)$$

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$$

$$\Rightarrow\ \psi_{2,1,1}\,(x,y,z) = \sqrt{\frac{8}{\mathrm{L_x L_y L_z}}}\,\sin\left(\frac{2\pi x}{L_x}\right)\,\sin\left(\frac{1\pi y}{L_y}\right)\,\sin\left(\frac{1\pi z}{L_z}\right)$$







3D PARTICLE IN A BOX: DEGENERACY

- Assume a cubic box $L = L_x = L_y = L_z$

$$\psi(x,y,z) = \sqrt{\frac{8}{L^3}}\,\sin\left(\frac{n_x \pi x}{L}\right)\,\sin\left(\frac{n_y \pi y}{L}\right)\,\sin\left(\frac{n_z \pi z}{L}\right)$$

- **Different wave function** solutions have **same energy**

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

- If such same energies occur we call this **degeneracy**
- Wave functions: **degenerate** quantum states
- Energies: **degenerate** energy levels
- Example: the energy level $3E_{1,1,1}$ is 3-fold degenerate

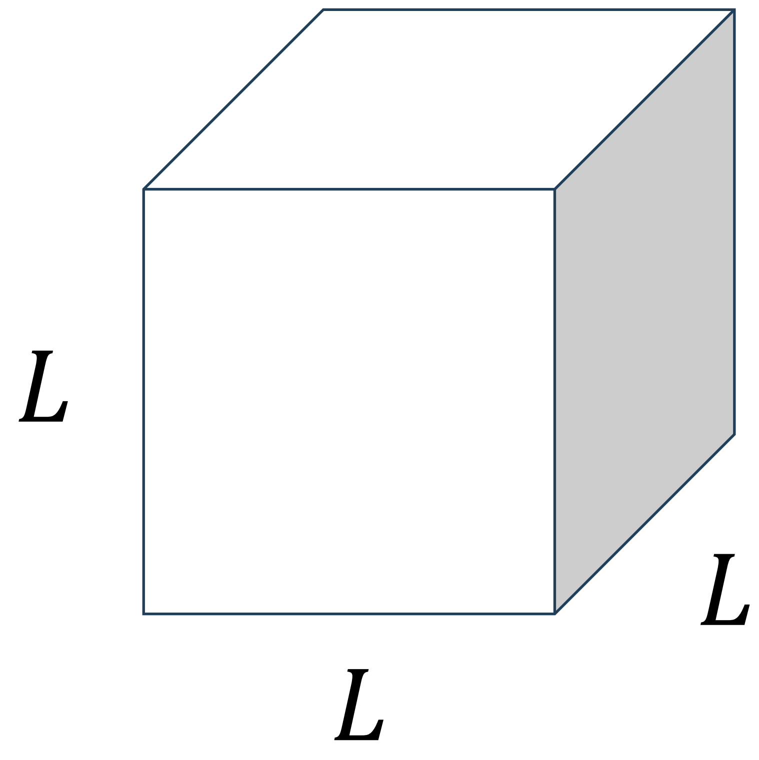
  with $\psi_{2,2,1}$, $\psi_{2,1,2}$, and $\psi_{1,2,2}$ having same energy

$\frac{14}{3}E_{1,1,1}$	(3,2,1), (3,1,2), (2,3,1), (1,3,2), (2,1,3),(1,2,3)
$4E_{1,1,1}$	(2,2,2)
$\frac{11}{3}E_{1,1,1}$	(3,1,1), (1,3,1), (1,1,3)
$3E_{1,1,1}$	(2,2,1), (2,1,2), (1,2,2)
$2E_{1,1,1}$	(2,1,1), (1,2,1), (1,1,2)
$E_{1,1,1}$	(1,1,1)
0	

3D PARTICLE IN A BOX: SYMMETRY → DEGENERACY

- Assume a cubic box $L = L_x = L_y = L_z$

$$\psi(x,y,z) = \sqrt{\frac{8}{L^3}}\,\sin\left(\frac{n_x \pi x}{L}\right)\,\sin\left(\frac{n_y \pi y}{L}\right)\,\sin\left(\frac{n_z \pi z}{L}\right)$$

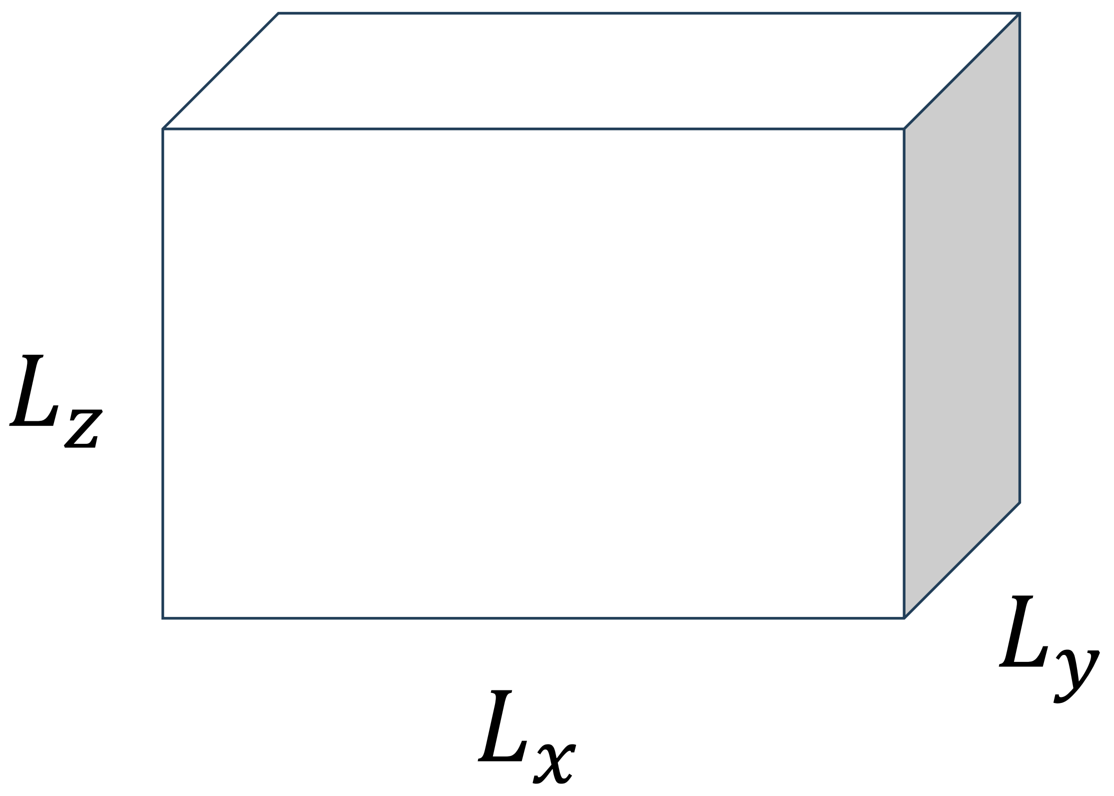


- Different wave function solutions have **same energy**

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

- **Symmetry of a cubic box** results in same situation if box is rotated over 90 degrees
- Introducing **asymmetry lifts the degeneracy**

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$$



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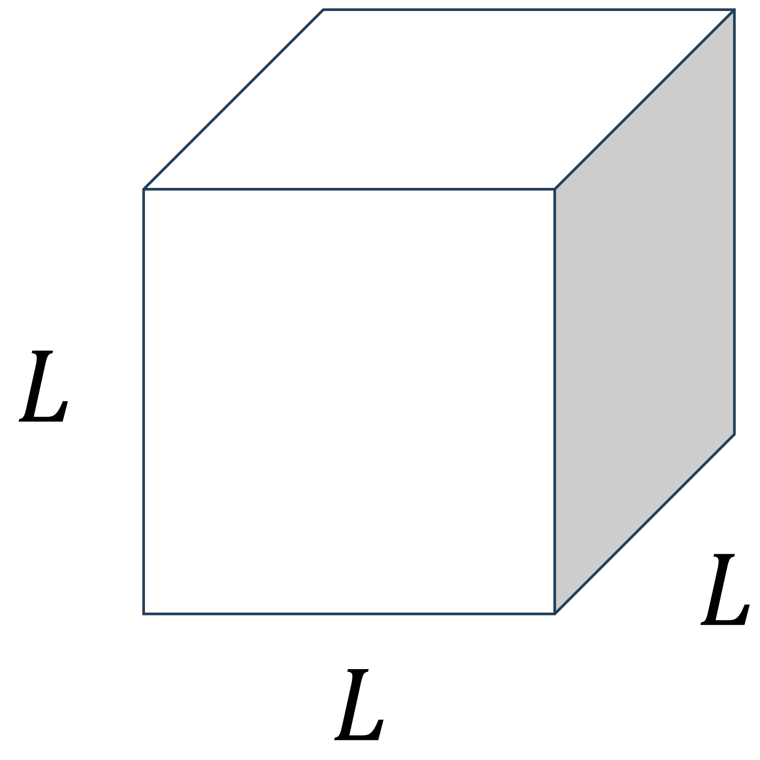

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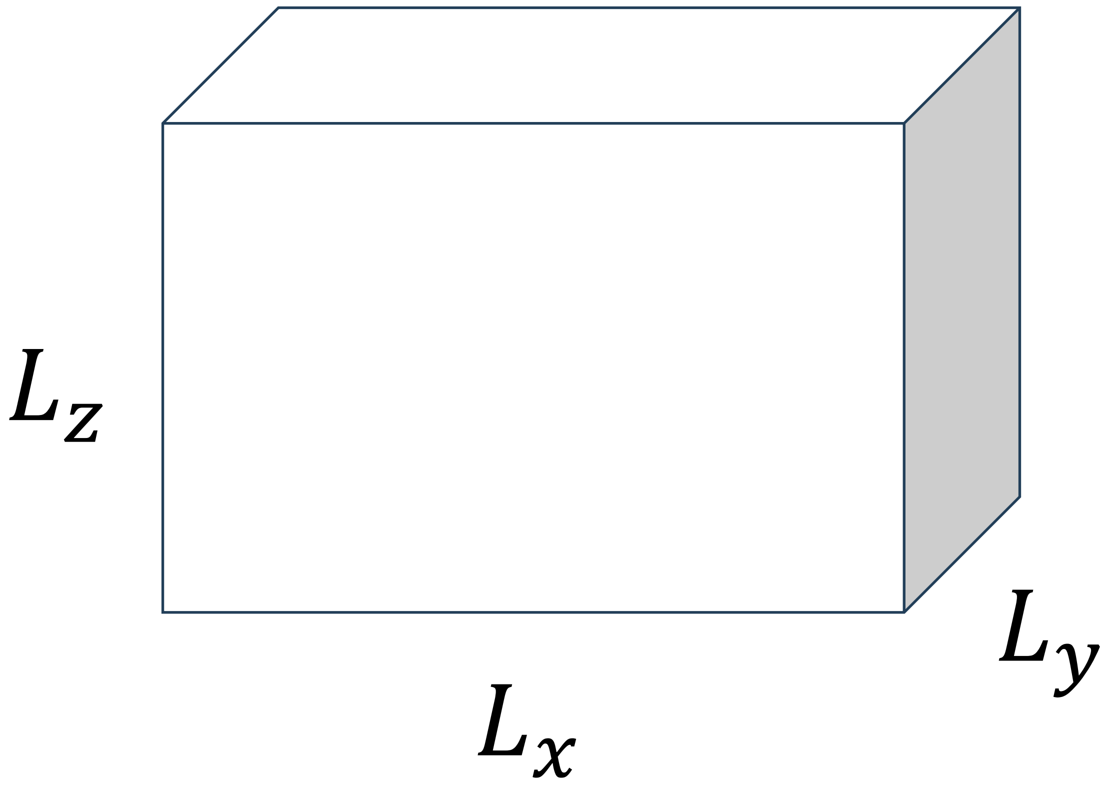


Quantum model of the Hydrogen atom

QUANTUM MODEL OF THE HYDROGEN ATOM

- 3D time-independent Schrodinger equation:

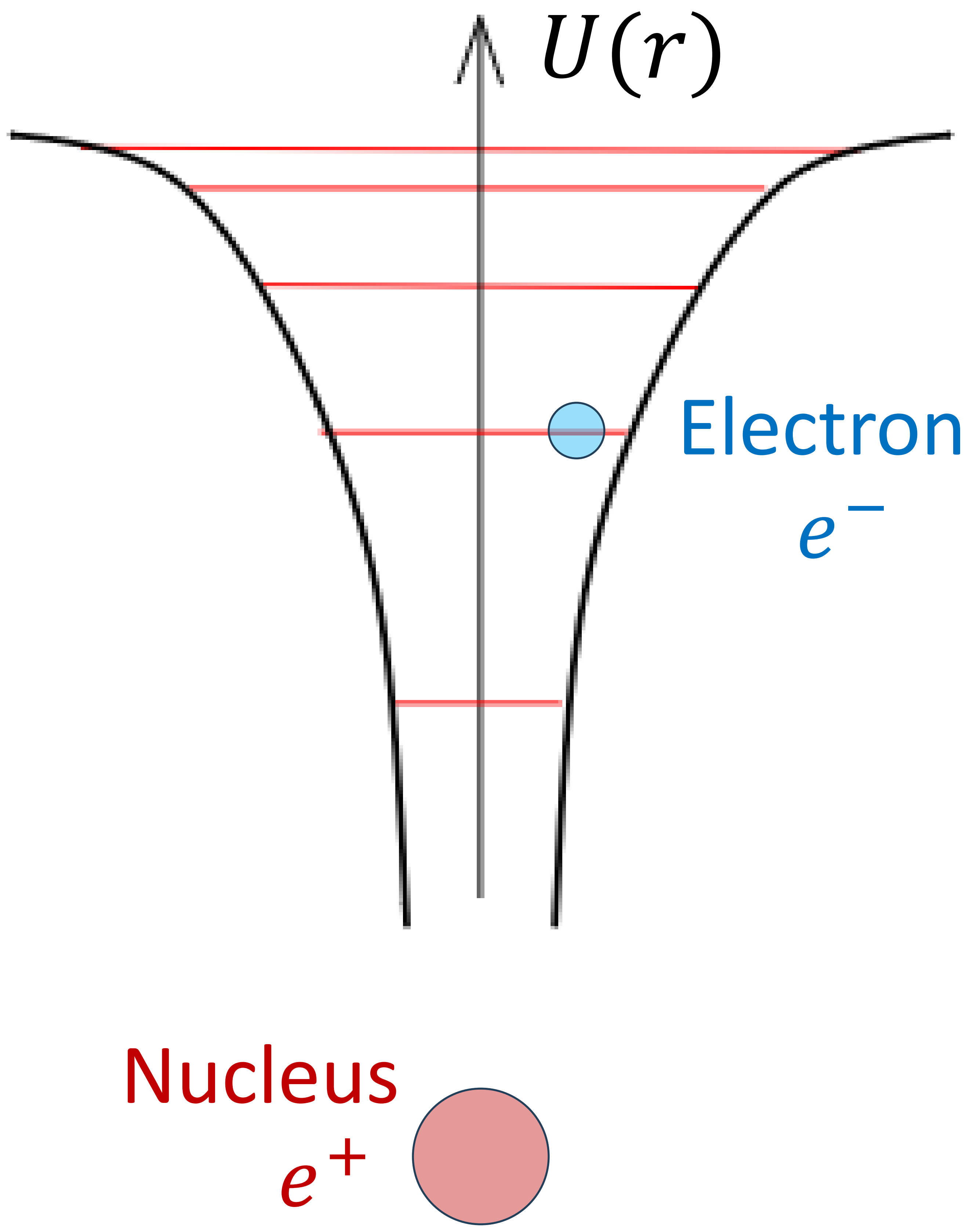
$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+U\,\psi=E\,\psi$$

- Use the Coulomb interaction between proton and electron:

$$U(r)=-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

- Spherical symmetry,
- convert to spherical coordinates:

$$\psi(x,y,z)\ \rightarrow\ \psi(r,\theta,\phi)$$



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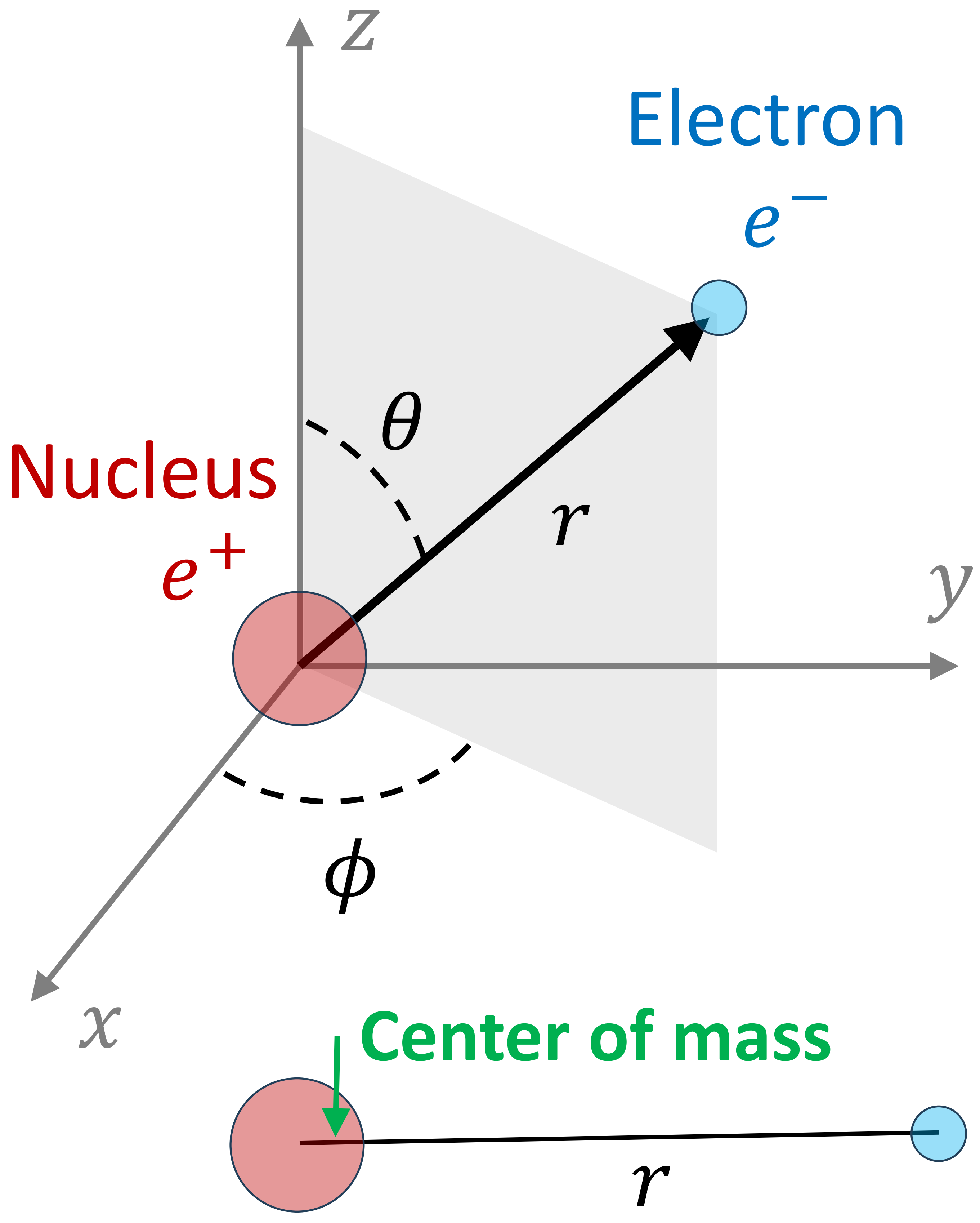
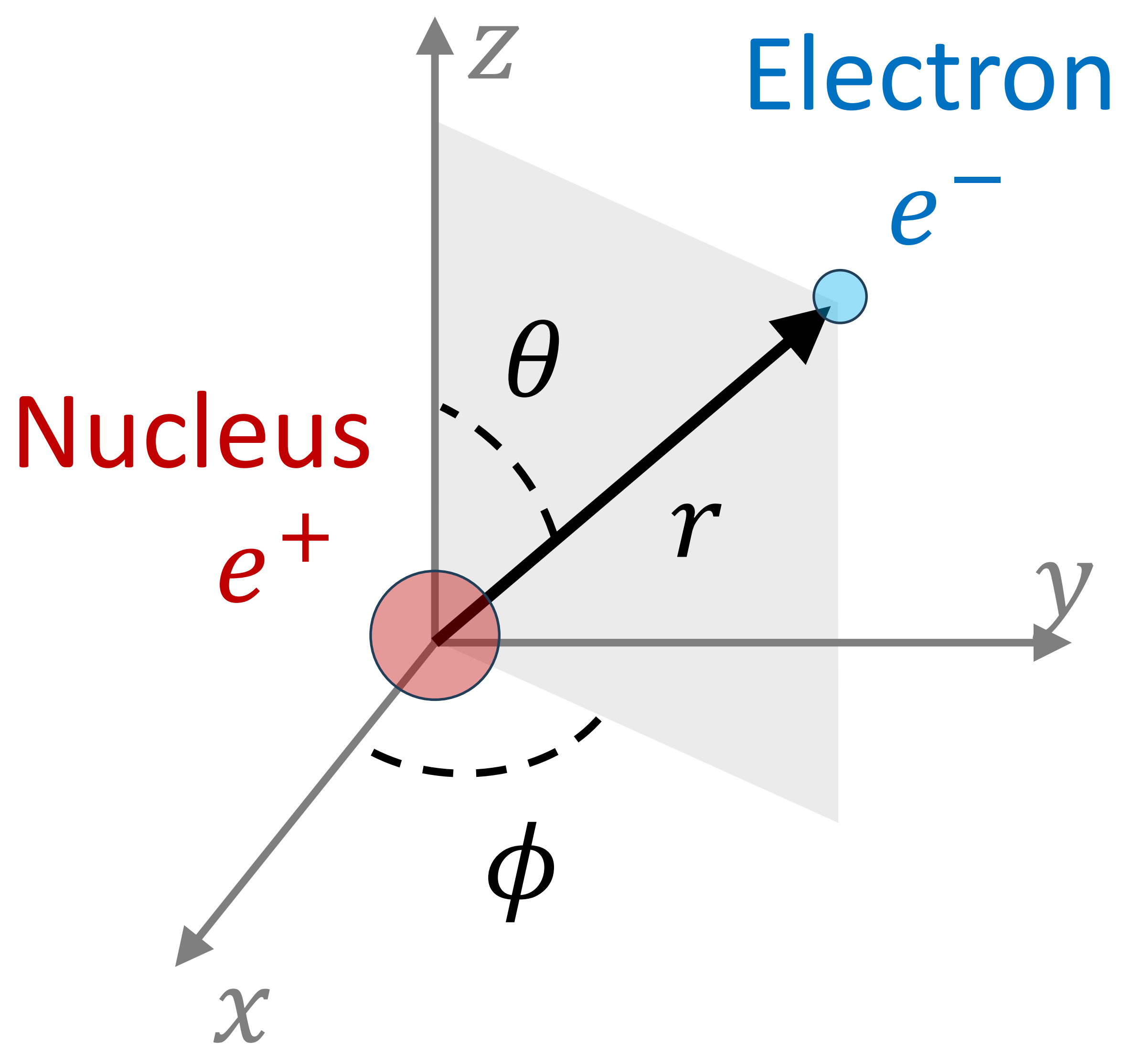


Reduced mass $\mu=\frac{m_e m_n}{m_e+m_n}\approx m_e$

HYDROGEN ATOM: SEPARATION OF VARIABLES

- 3D time-independent Schrodinger equation:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}\right)+U\,\psi=E\,\psi$$



- Separation of the variables (spherical coordinates):

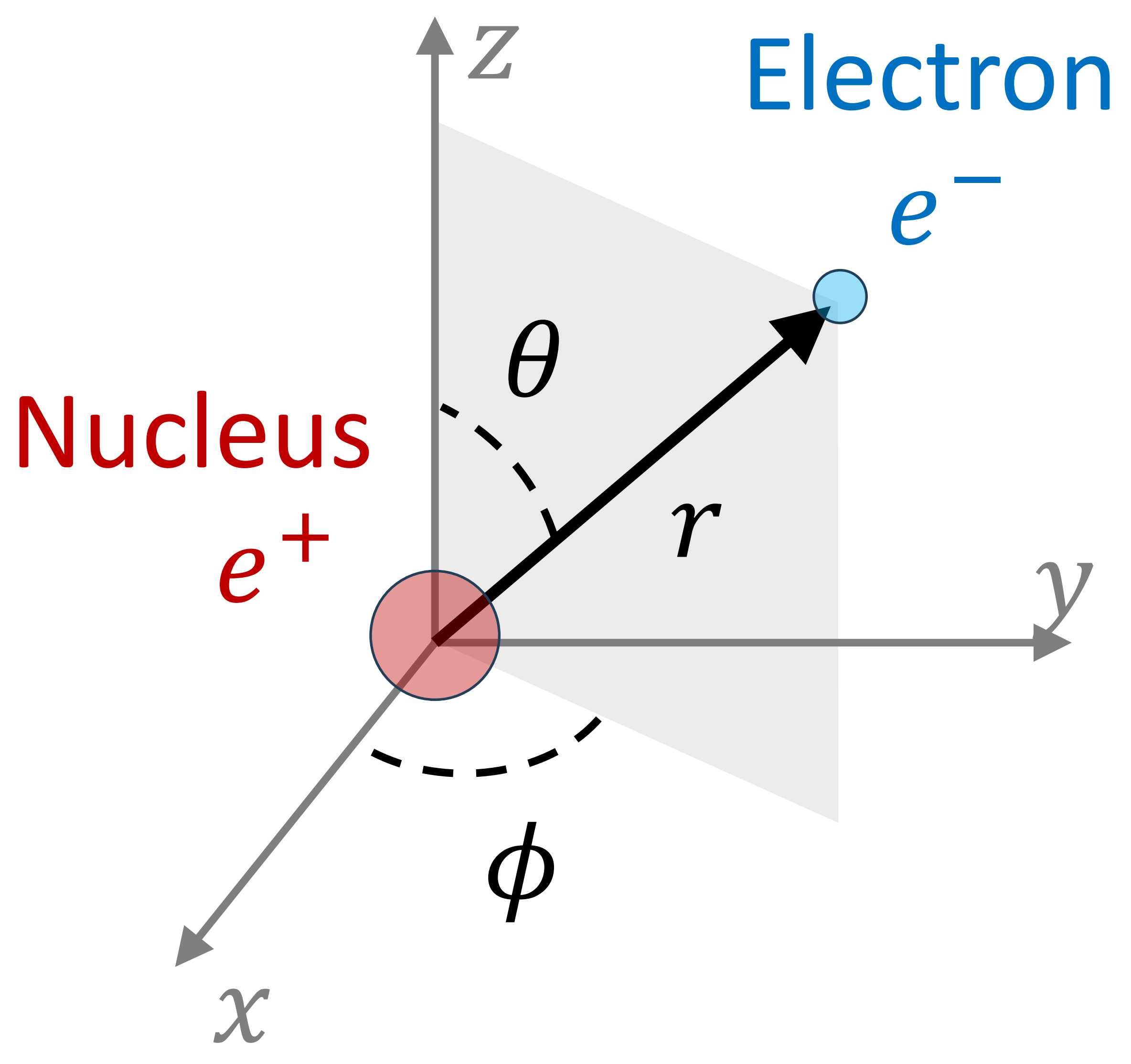
$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$$

$$\Rightarrow \left\{\begin{array}{l} -\dfrac{\hbar^2}{2\mu\, r^2}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right)+\left(\dfrac{\hbar^2 l(l+1)}{2\mu r^2}+U(r)\right)R(r)=E\,R(r) \\ -\dfrac{1}{\sin\theta}\left(\sin\theta\,\dfrac{d\Theta(\theta)}{d\theta}\right)+\left(l(l+1)-\dfrac{m_l^2}{\sin^2\theta}\right)\Theta(\theta)=0 \\ \dfrac{d^2\Phi(\phi)}{d\phi^2}+m_l^2\Phi(\phi)=0 \end{array}\right.$$

HYDROGEN ATOM: SEPARATION OF VARIABLES

- Separation of the variables (spherical coordinates):

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$



$$\begin{cases} -\dfrac{\hbar^2}{2\mu\, r^2}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right) + \left(\dfrac{\hbar^2 l(l+1)}{2\mu r^2} + U(r)\right)R(r) = E\,R(r) \\ -\dfrac{1}{\sin\theta}\left(\sin\theta\,\dfrac{\mathrm{d}\Theta(\theta)}{d\theta}\right) + \left(l(l+1) - \dfrac{m_l^2}{\sin^2\theta}\right)\Theta(\theta) = 0 \\ \dfrac{\mathrm{d}^2\Phi(\phi)}{d\phi^2} + m_l^2\Phi(\phi) = 0 \end{cases}$$

- Solutions?
- Let's first look at the conditions

$R(r \to \infty) \to 0$
$\Phi(\phi)$ and $\Theta(\theta)$ finite everywhere
$\Phi(\phi + 2\pi) = \Phi(\phi)$

HYDROGEN ATOM: RADIAL EQUATION

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Solution of the radial part:

$$-\frac{\hbar^2}{2\mu\, r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\frac{\hbar^2 l(l+1)}{2\mu r^2} + U(r)\right)R(r) = E\, R(r)$$

- Condition: $R(r \to \infty) \to 0$
- Solutions

$$R_{n,l}(r) \propto \left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right) \times e^{-\frac{r}{na_0}}$$

HYDROGEN ATOM: RADIAL EQUATION

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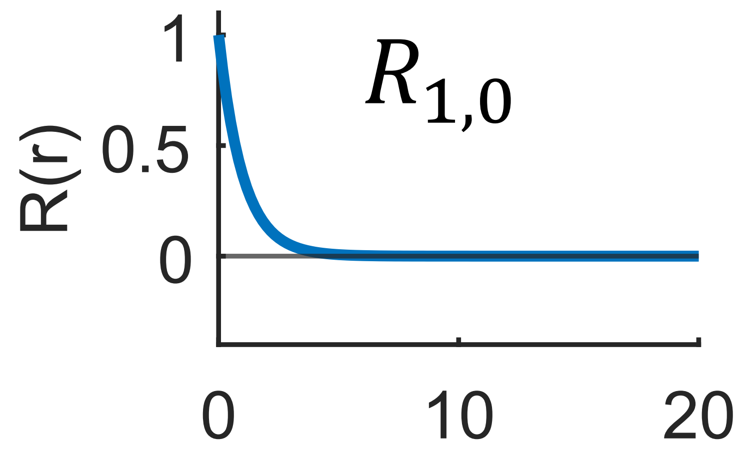
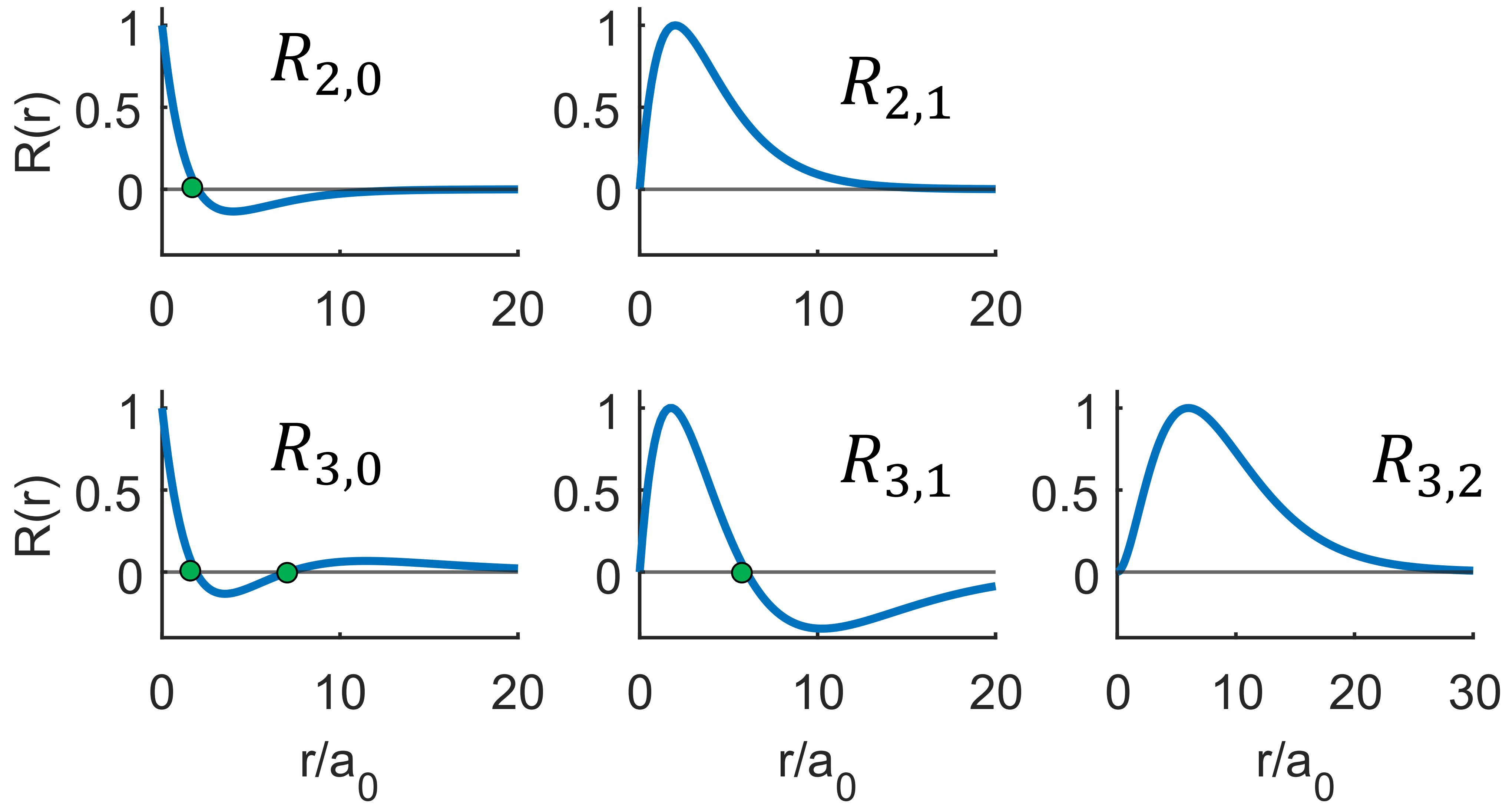
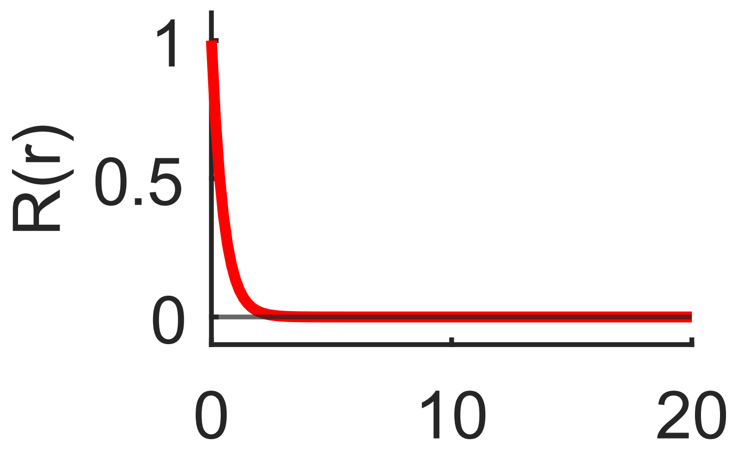


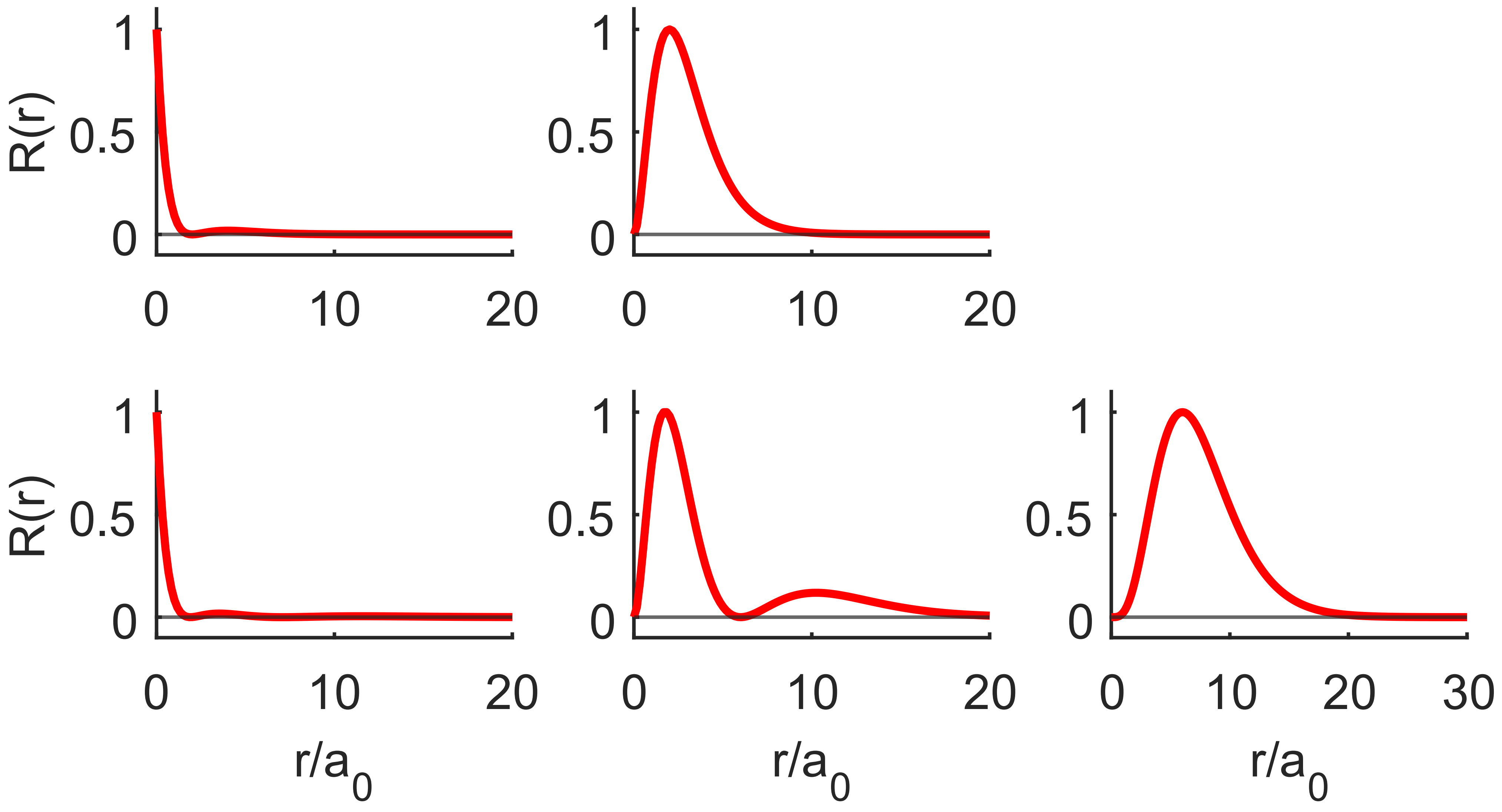
$R_{n,l}(r)$

$n - l - 1$ zeros





$|R_{n,l}(r)|^2$

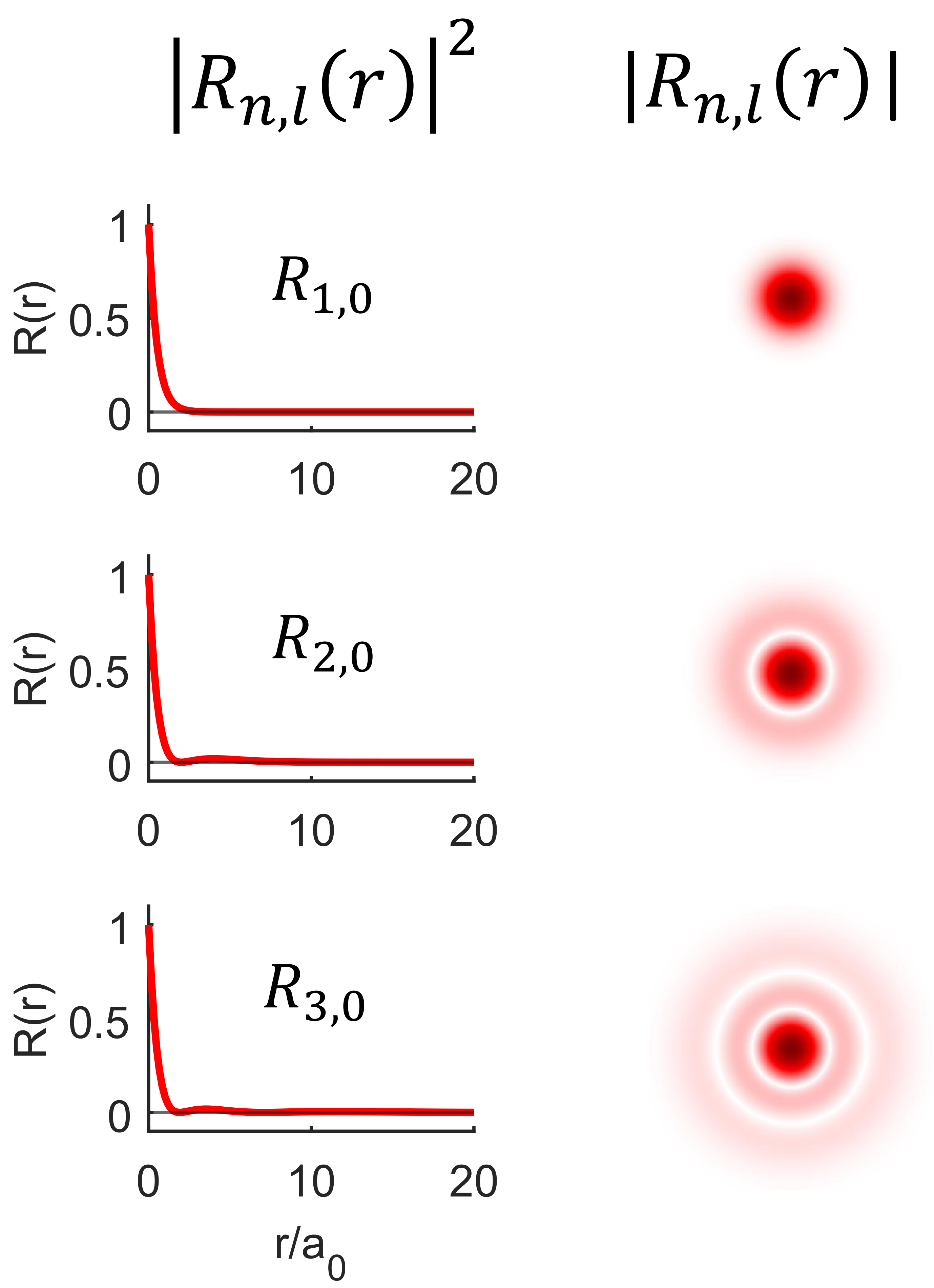


HYDROGEN ATOM: RADIAL EQUATION

$$R_{n,l}(r) \propto \left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right) \times e^{-\frac{r}{na_0}}$$

- **Principal quantum number:** $n = 1, 2, 3, \ldots, \infty$
- **Orbital quantum number:** $l = 0, 1, 2, \ldots, n-1$
- Radial part has $n - l - 1$ zeros in probability
- Energy is defined by the radial part:

$$E_n = -\frac{1}{(4\pi\epsilon_0)^2}\frac{\mu e^4}{2n^2\hbar^2} = -\frac{13.6\text{ eV}}{n^2}$$



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HYDROGEN ATOM: RADIUS OF S-ORBITAL

- Is the electron likely to be at the nucleus?
- Expected radius of the electron?
- Consider the ground state:

$$\psi_{1,0,0}(r) \propto \sqrt{\frac{1}{\pi a_0^3}}\, e^{-\frac{r}{1 a_0}}$$

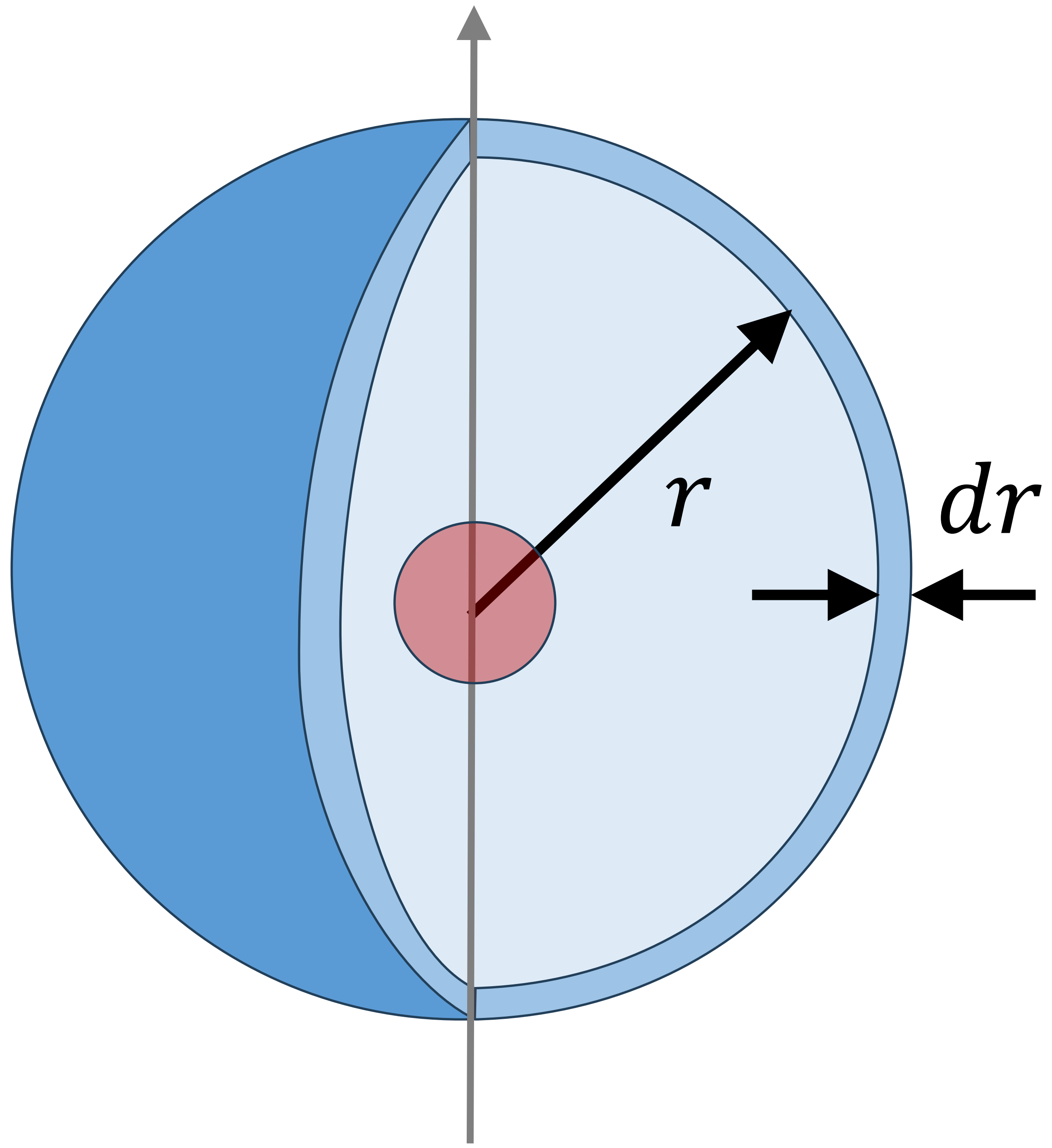
- Probability to have radius r :

$$P(r)\, dV = \left|\psi_{1,0,0}(r)\right|^2 dV$$

- Volume infinitesimal thin shell radius r :

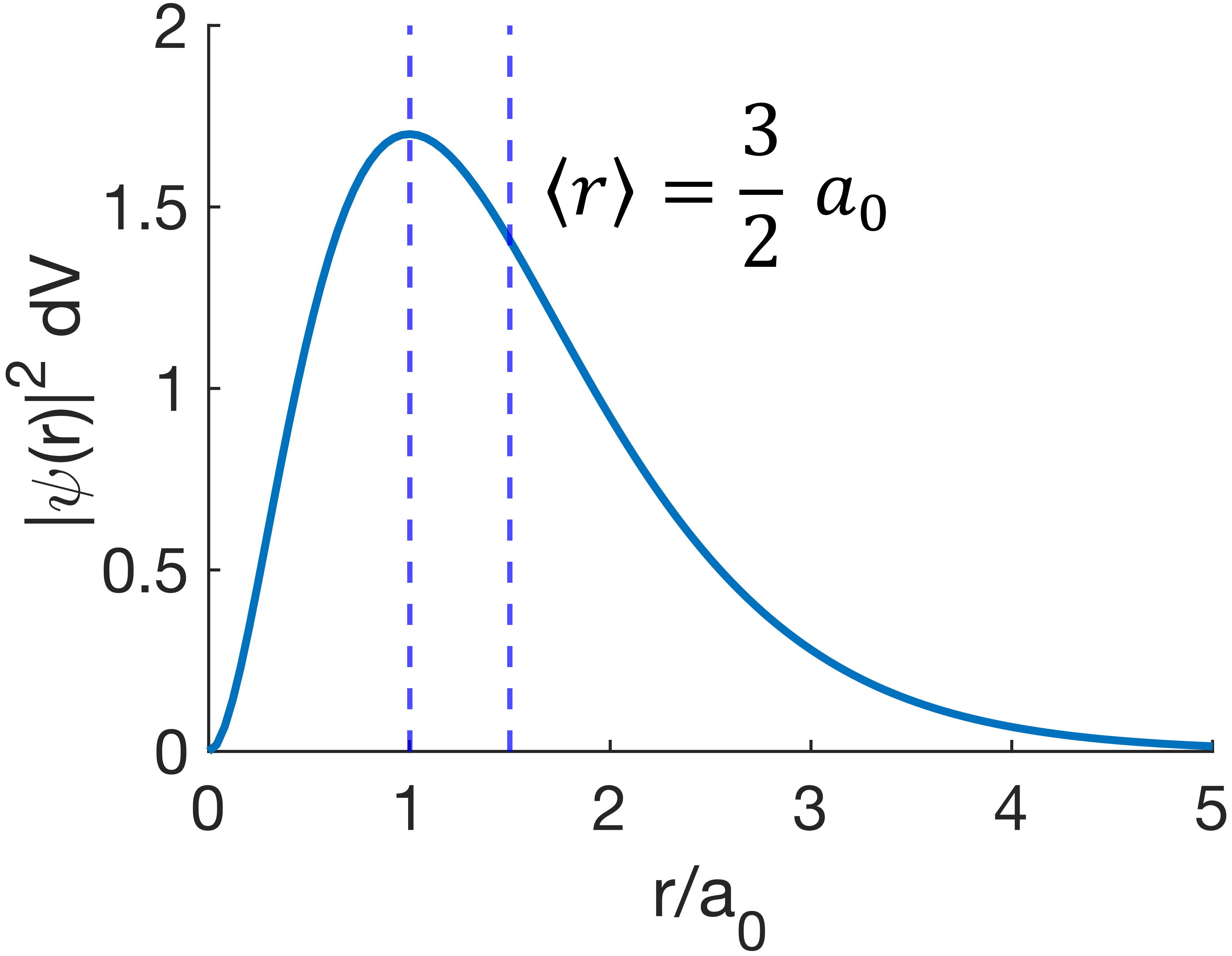
$$V_{\text{shell}} = 4\pi\, r^2 dr$$

$$P(r)\, dV = \left|\psi_{1,0,0}(r)\right|^2 4\pi r^2 dr$$

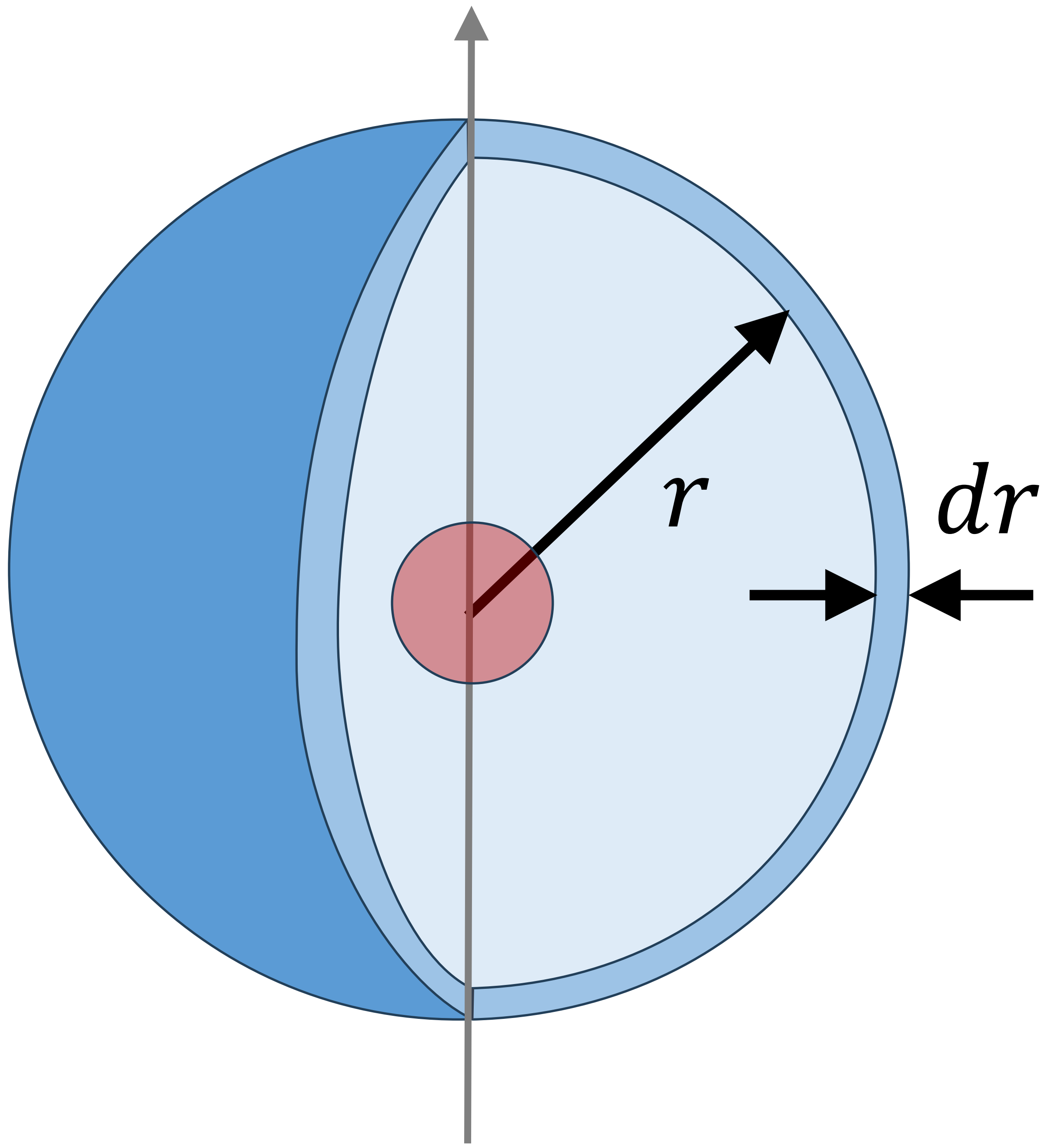


HYDROGEN ATOM: RADIUS OF S-ORBITAL

$$r_{\max} = a_0$$



$$P(r)\, dV = \left|\psi_{1,0,0}(r)\right|^2 4\pi r^2 dr$$



HYDROGEN ATOM: RADIUS OF S-ORBITAL

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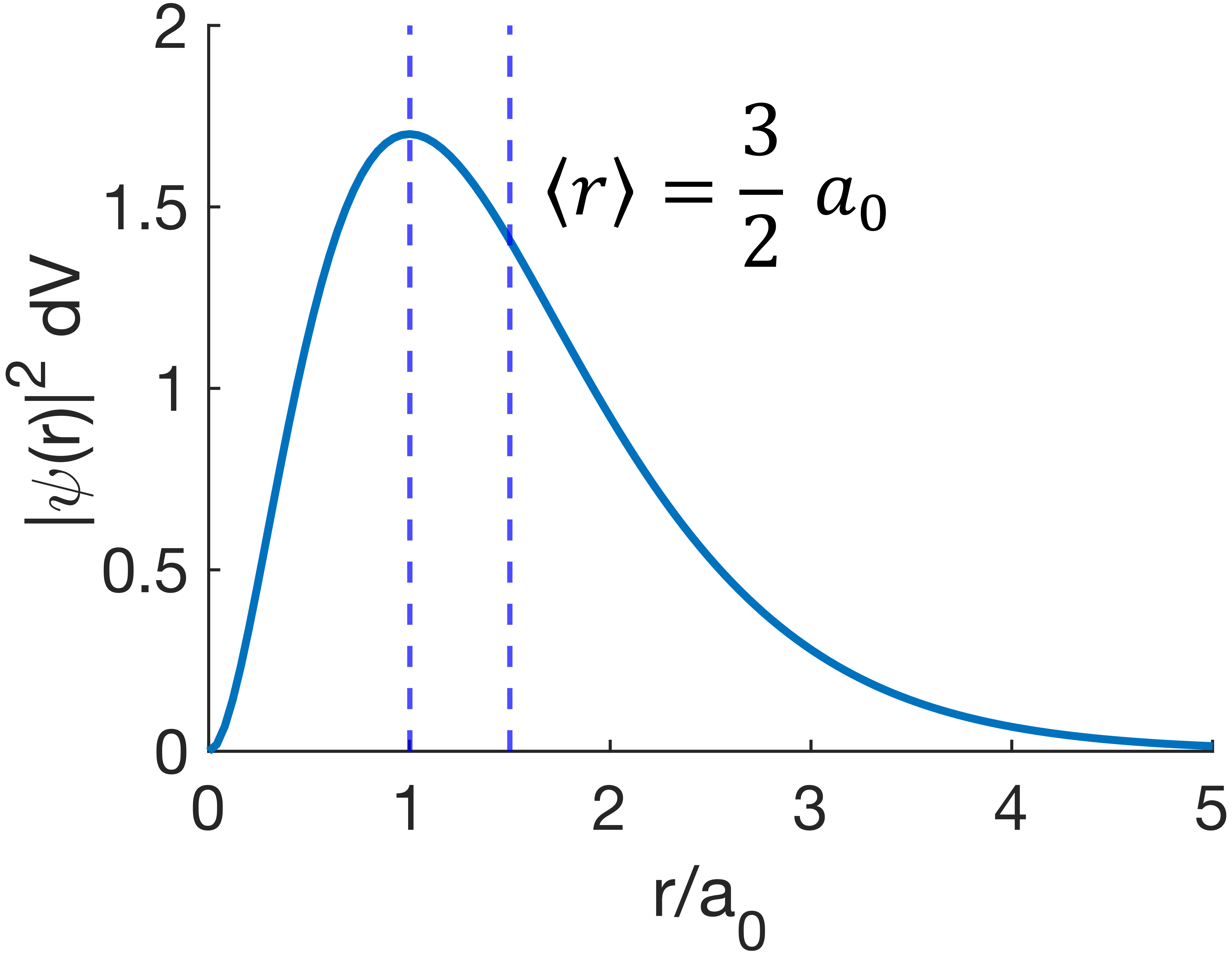


- Expectation value for radius r

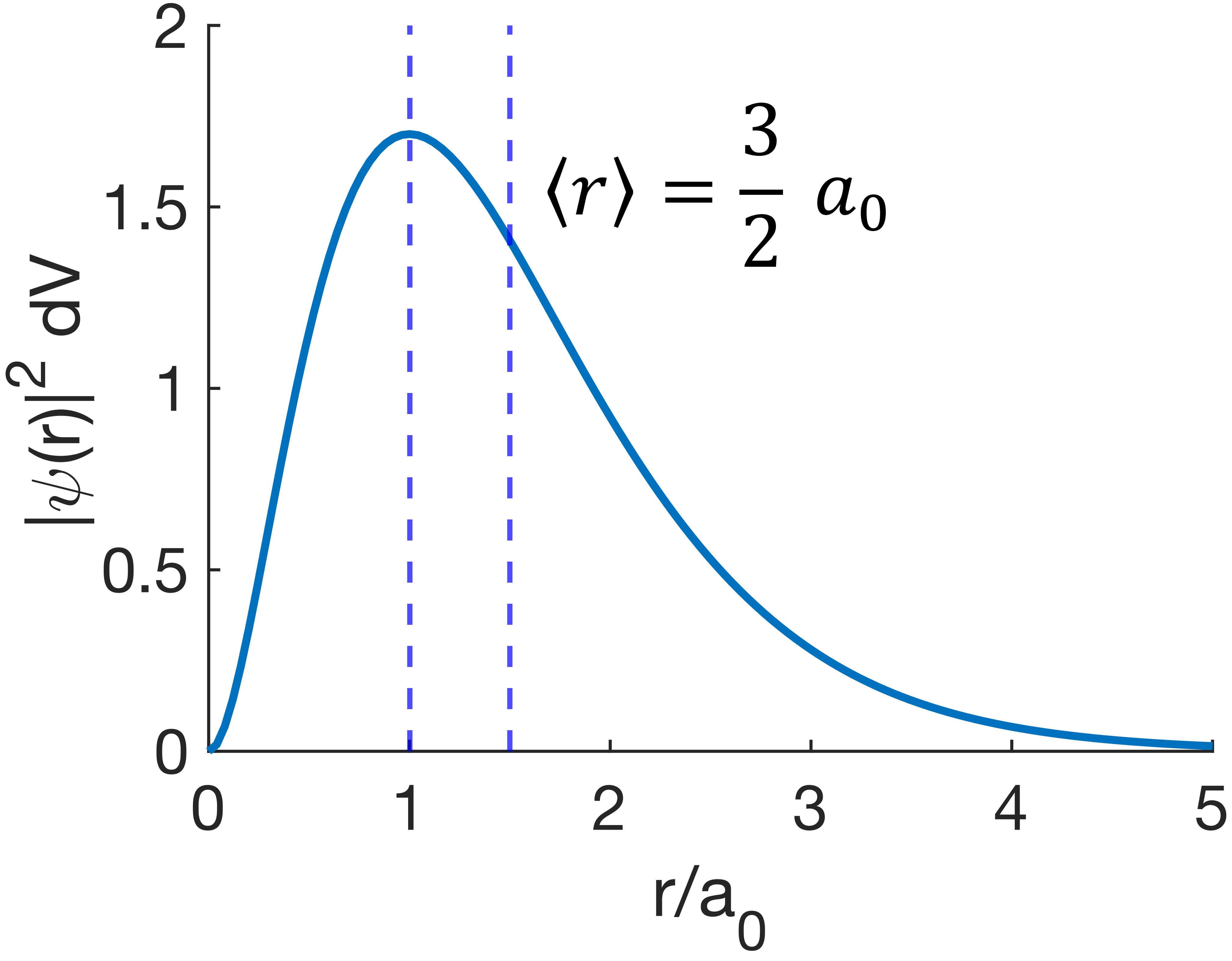
$$\langle r \rangle = \int_0^\infty r\, \left|\psi_{1,0,0}(r)\right|^2 4\pi\, r^2 dr$$

$$= \frac{1}{\pi a_0^3} 4\pi \int_0^\infty r^3 e^{-\frac{2r}{a_0}} dr$$

$$= \frac{4}{a_0^3} \frac{3!}{\left(\frac{2}{a_0}\right)^4} = \frac{3}{2} a_0$$

HYDROGEN ATOM: RADIUS OF S-ORBITAL

$r_{\max} = a_0$



$\langle r \rangle = \frac{3}{2}\, a_0$

- Most likely radius r

$$0 = \frac{d}{dr}\left(\left|\psi_{1,0,0}(r)\right|^2 4\pi\, r^2\right)$$

$$= \frac{d}{dr}\left(\frac{4}{a_0^3}\, r^2 e^{-\frac{2r}{a_0}}\right)$$

$$= \frac{4}{a_0^3}\left(2 - \frac{2r}{a_0}\right) r e^{-\frac{2r}{a_0}}$$

$$\Rightarrow \quad 2 = \frac{2r}{a_0} \quad \Rightarrow \quad \mathrm{r}_{\max} = \mathrm{a}_0$$

HYDROGEN ATOM: ANGULAR MOMENTUM

- Quantization of angular momentum

$$L = \sqrt{l(l+1)}\,\hbar \quad \text{with} \quad l = 0, 1, \dots, n-1$$

- Different from Bohr's: $l = 0 \Rightarrow L = 0$

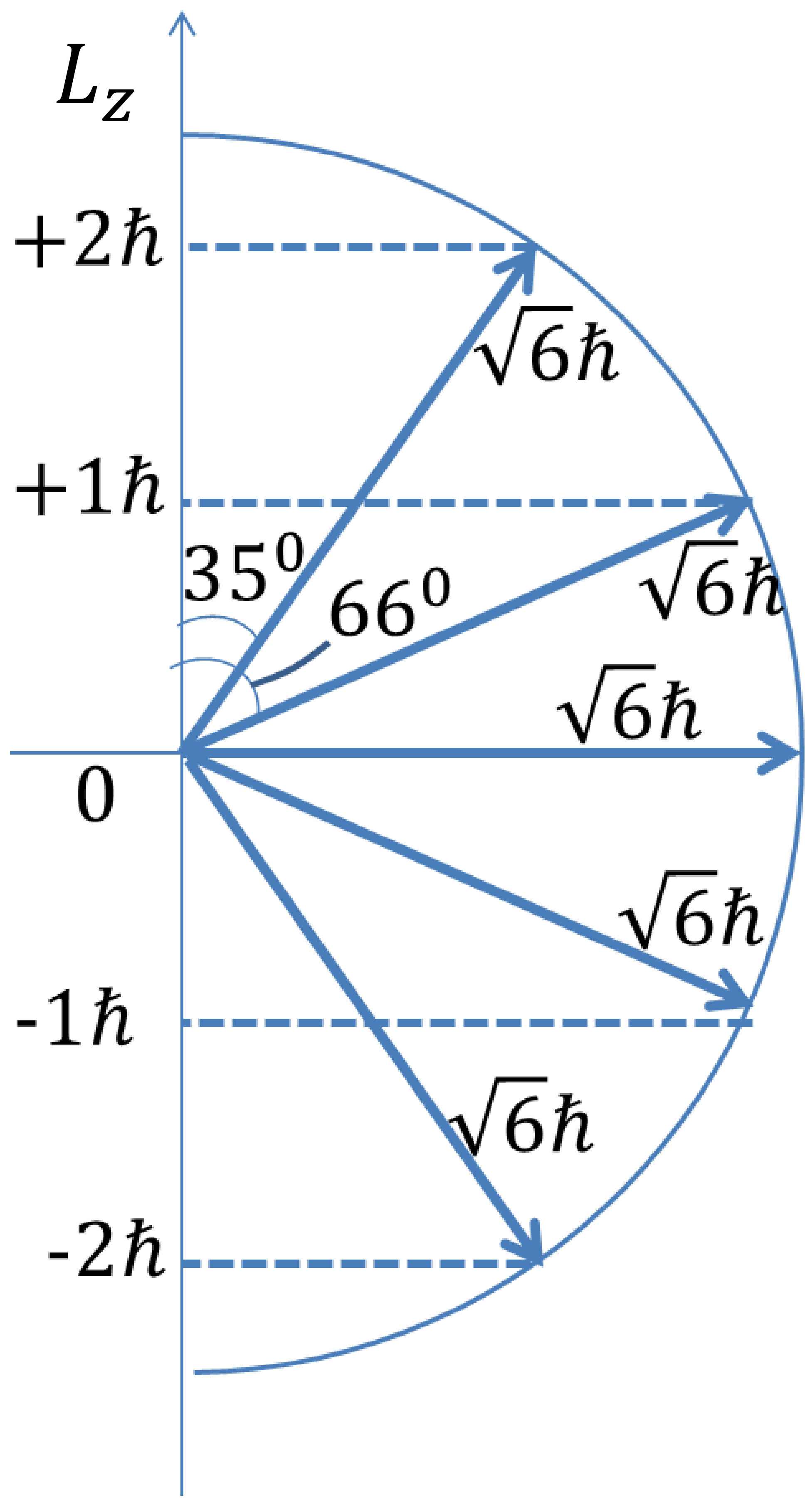
- Condition $\Phi(\phi) = \Phi(\phi + 2\pi)$ gives:

$$L_z = m_l \hbar \quad \text{with} \quad m_l = 0, \pm 1, \dots, \pm l$$

Quantum numbers so far:

n: principal quantum number
l: orbital quantum number
m_l: magnetic quantum number

Graphical representation relation L_z and L



HYDROGEN ATOM: ANGULAR SOLUTIONS

- Quantization of angular momentum

$$L = \sqrt{l(l+1)}\,\hbar \quad \text{with} \quad l = 0, 1, \dots, n-1$$

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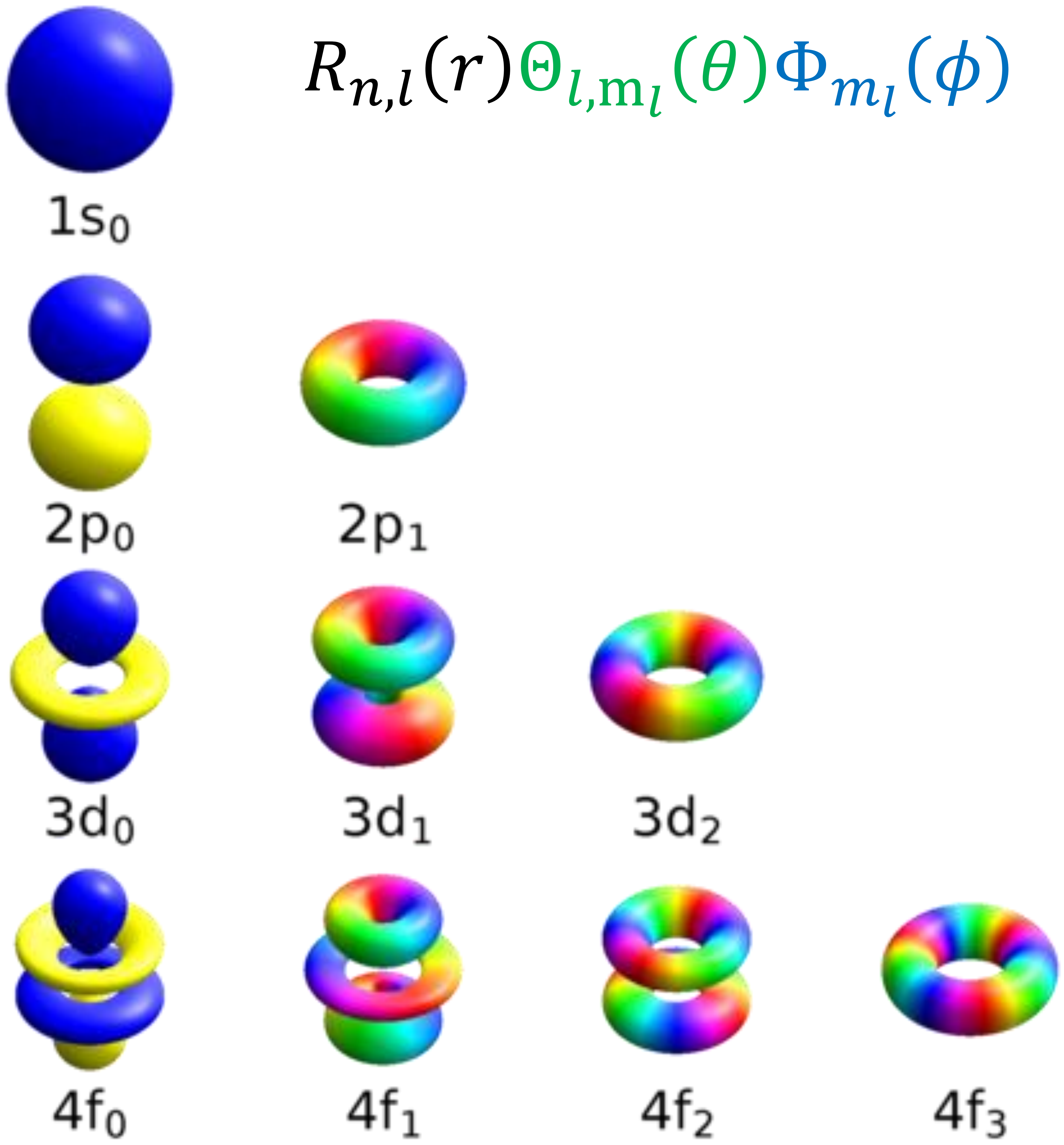
Quantum numbers so far:

n: principal quantum number
l: orbital quantum number
m_l: magnetic quantum number

$$R_{n,l}(r)\Theta_{l,m_l}(\theta)\Phi_{m_l}(\phi)$$



HYDROGEN ATOM: ANGULAR EQUATIONS

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Solving the $\Phi(\phi)$ part:

$$\frac{\mathrm{d}^2\Phi(\phi)}{d\phi^2} + m_l^2\Phi(\phi) = 0$$

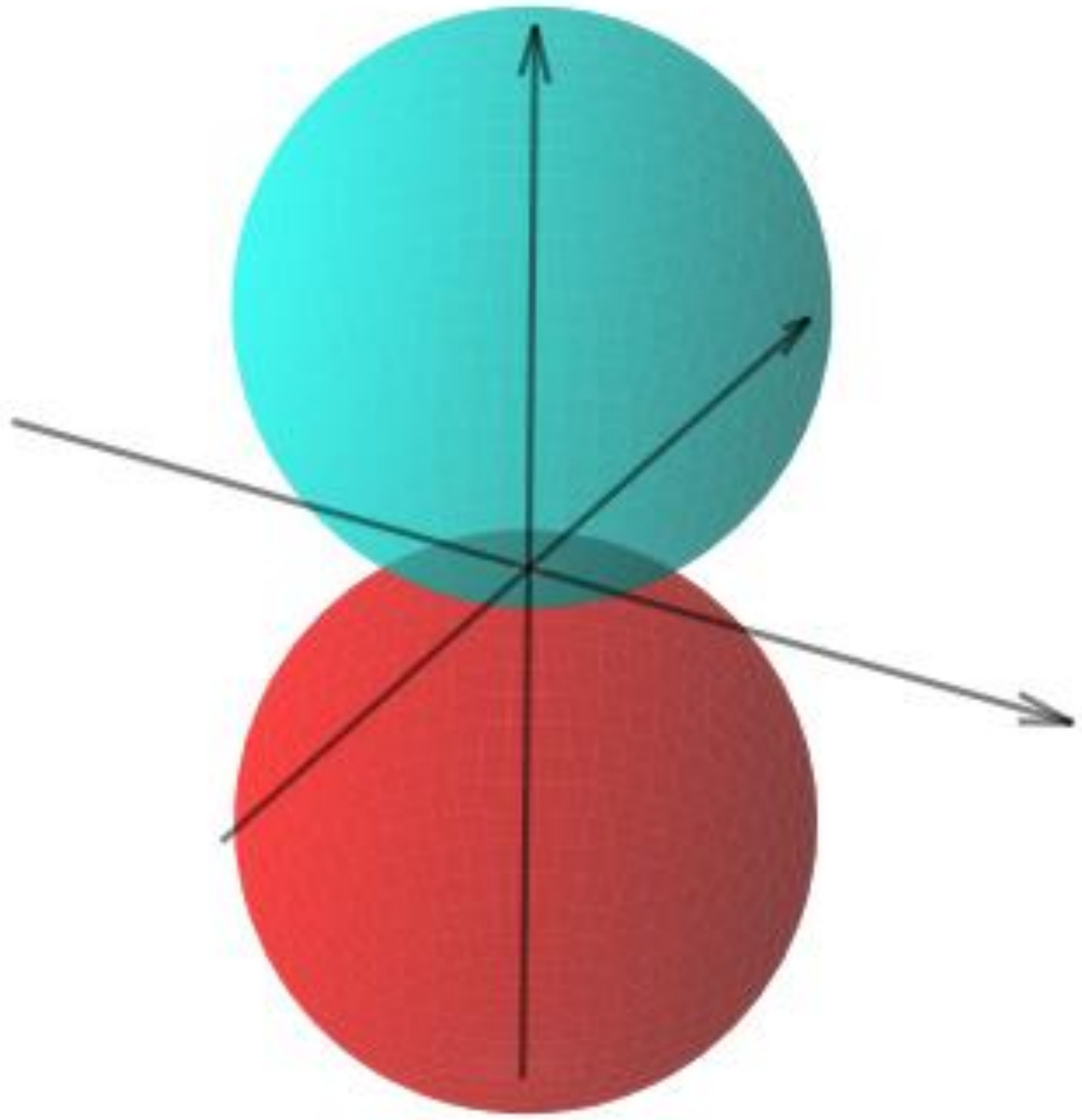
- Conditions:

  $\Phi(\phi)$ finite everywhere, and

  $\Phi(\phi + 2\pi) = \Phi(\phi)$

- Solution: a **complex phase factor**

$$\boxed{\Phi_{\mathrm{m_l}}(\phi) \propto e^{\,i m_l \phi}}$$



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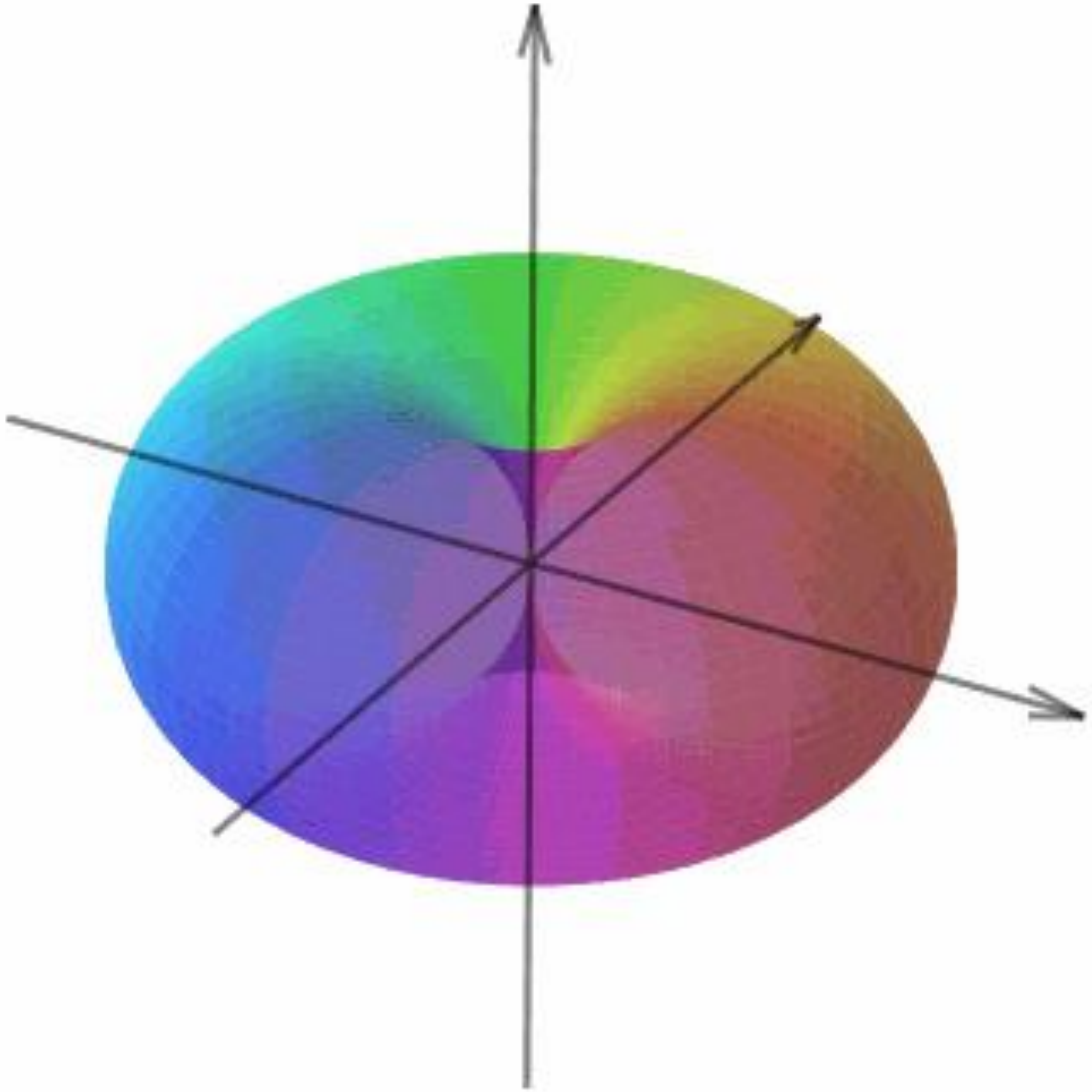
$$\Phi_{m_l}(\phi) \propto e^{im_l\phi}$$



HYDROGEN ATOM: SYMMETRY & SUPERPOSITION

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

- Solution: a **complex phase factor**

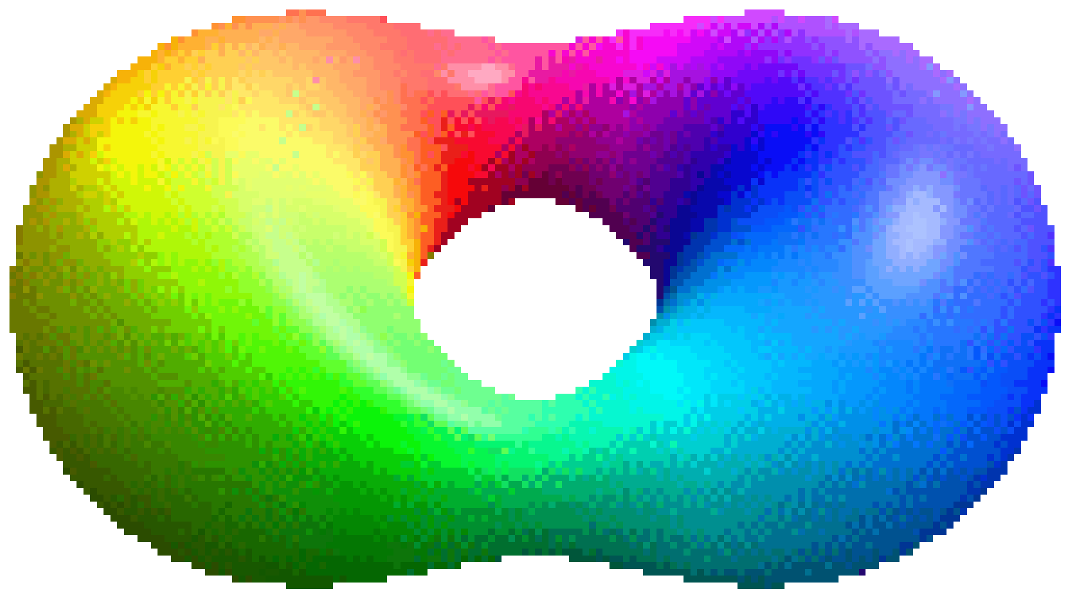
$$\Phi_{\mathrm{m_l}}(\phi) \propto e^{im_l\phi}$$

- Solution with P0

$$2\mathrm{Px} - \mathrm{orbital}$$

$$R(r)\Theta(\theta) \times (e^{im_l\phi} + e^{-im_l\phi})$$

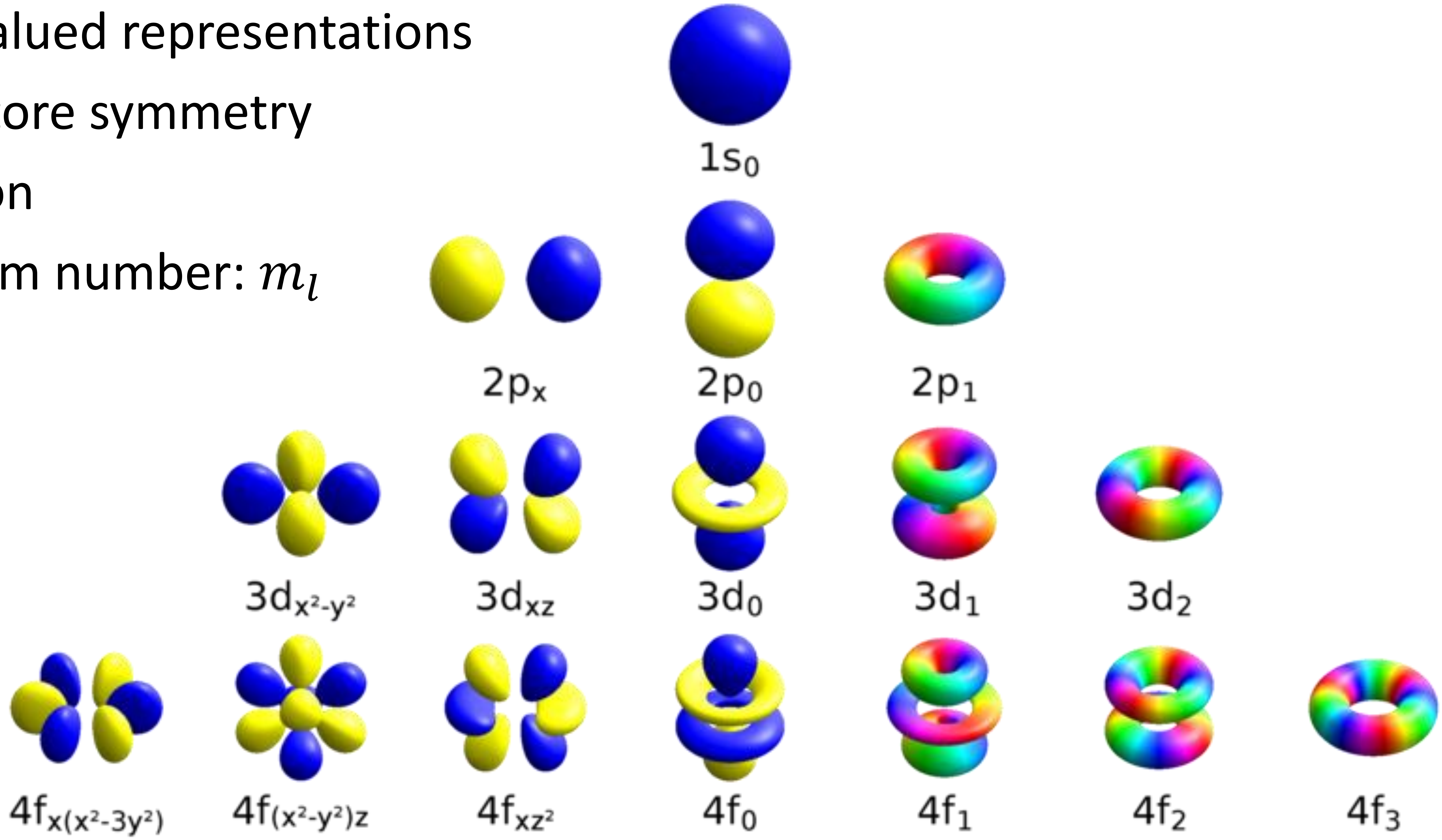
0.99|2p1⟩ + 0.11|2p-1⟩



0.78|2px⟩ + 0.63i|2py⟩

HYDROGEN ATOM: SUMMARY

- More real-valued representations
- Partially restore symmetry
- Superposition
- Lose quantum number: m_l



HYDROGEN ATOM: QUANTIZATION AND NAMING

- Quantization of energy, linear momentum and radius
- Quantization of orbital/azimuthal momentum
- Energy E_n is degenerate: l and m_l do not influence it
- Expected radius/energy close to Bohr's model
- Labeling and naming conventions:

Principal quantum number	Shell name
$n = 1$	K shell
$n = 2$	L shell
$n = 3$	M shell
$n = 4$	N shell

Azimuthal quantum number	Subshell name
$l = 0$	S states
$l = 1$	P states
$l = 2$	D states
$l = 3$	F states