PHOT 222: Quantum Photonics
LECTURE 08

Michaël Barbier, Spring semester (2024-2025)

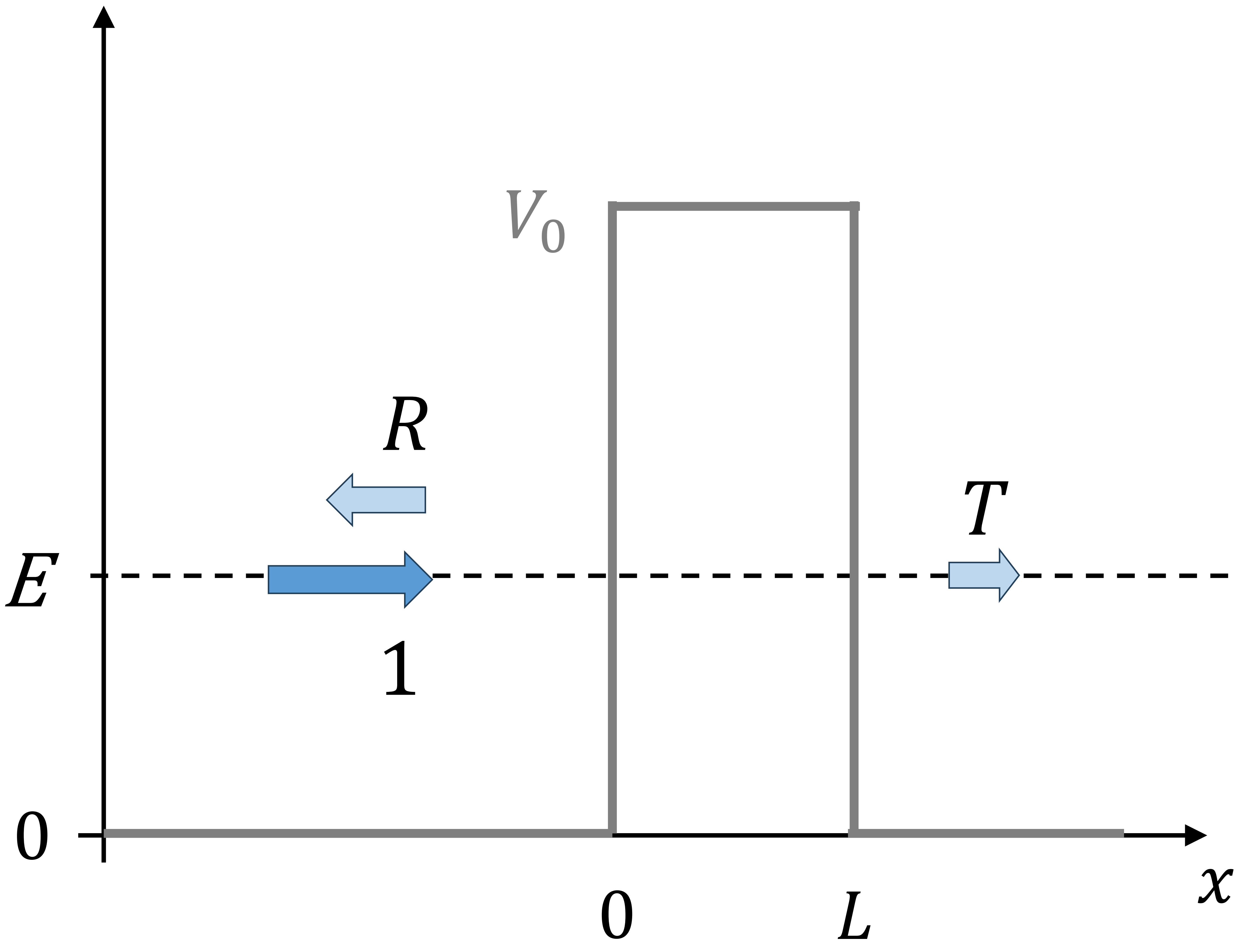
OVERVIEW OF THE COURSE

week	topic	Serway 9th	Young
Week 1	Relativity	Ch. 39	Ch. 37
Week 2	Waves and Particles	Ch. 40	Ch. 38-39
Week 3	Wave packets and Uncertainty	Ch. 40	Ch. 38-39
Week 4	The Schrödinger equation and Probability	Ch. 41	Ch. 39
Week 5	**Midterm exam 1**		
Week 6	Quantum particles in a potential	Ch. 41	Ch. 40
Week 7	Bayram		
Week 8	**Tunneling through a potential barrier**	**Ch. 41**	**Ch. 40**
Week 9	**Harmonic oscillator**	**Ch. 41**	**Ch. 40**
Week 10	**Midterm exam 2**		
Week 11	The hydrogen atom, absorption/emission spectra		
Week 12	Many-electron atoms & Pauli-exclusion principle		
Week 13	Atomic bonds and molecules		
Week 14	Crystalline materials and energy band structure		

Tunneling through a barrier

TUNNELING THROUGH A POTENTIAL BARRIER

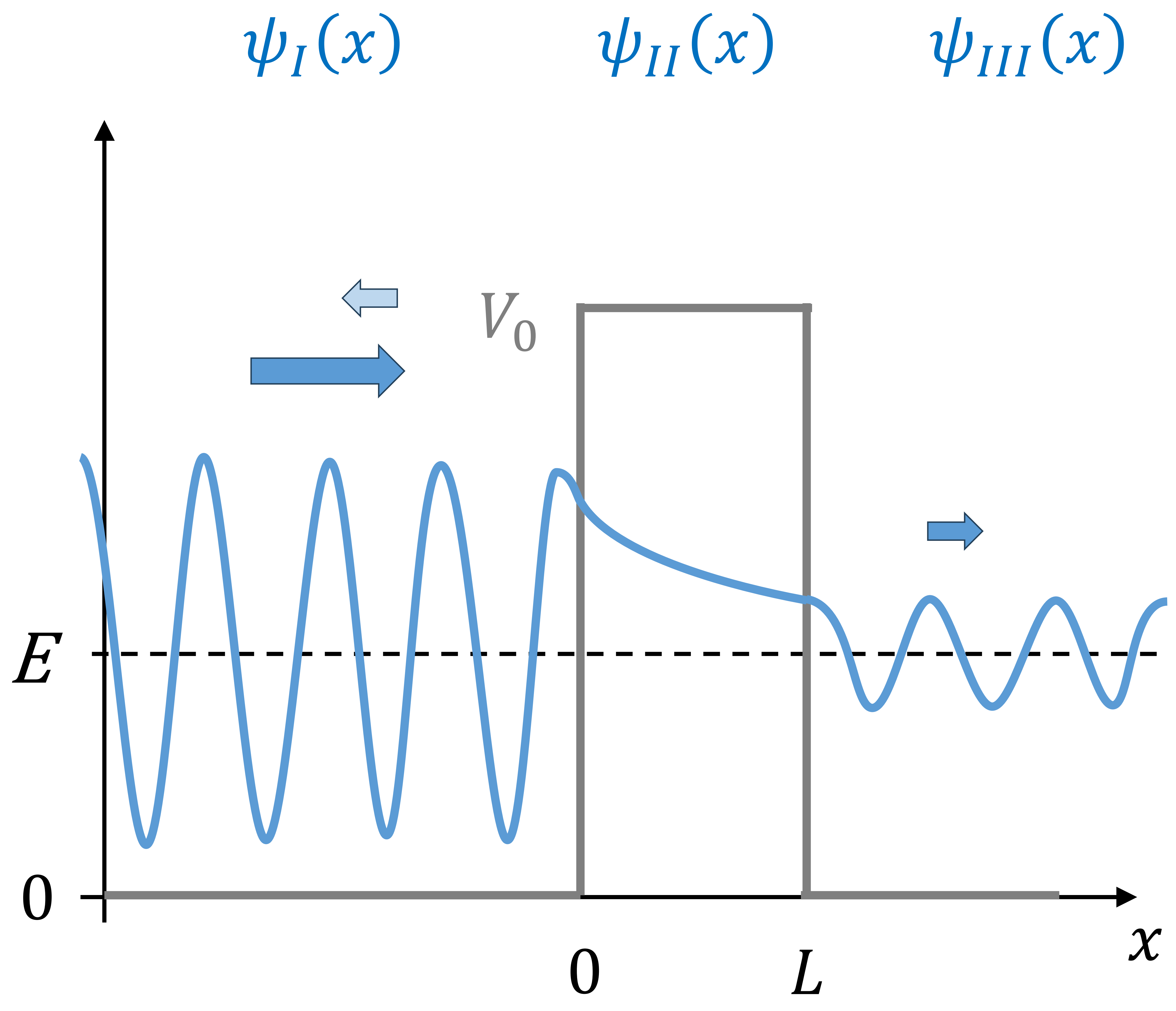
- Quantum particle can **tunnel** through a potential barrier
- **Classically** the particle cannot tunnel
- **Tunneling probability: Transmission** coefficient T
- **Reflection** $R \rightarrow \quad T + R = 1$



TUNNELING THROUGH A POTENTIAL BARRIER

Energy lower than barrier

- Wave function ψ **tunnels** through barrier
- **Evanescent waves** in barrier: exponentially decaying
- Reduced amplitude after barrier
- How to find the wave function?



TUNNELING THROUGH A POTENTIAL BARRIER

- Regions: I, II, III
- Wave function in regions:

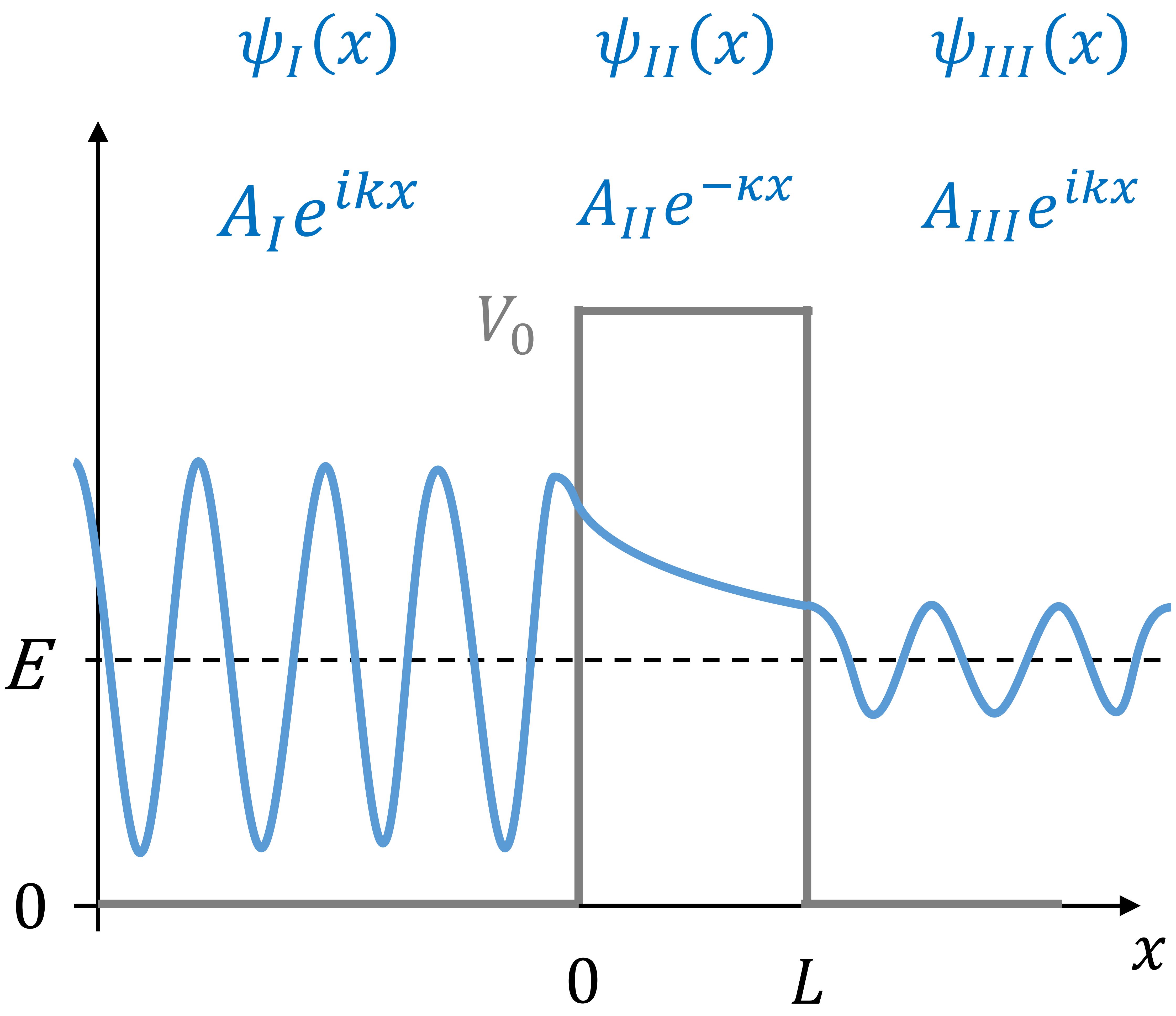
$$\psi_I(x) = A_I e^{ikx}$$
$$\psi_{II}(x) = A_{II} e^{-\kappa x}$$
$$\psi_{III}(x) = A_{III} e^{ikx}$$

- With wave numbers:

$$\kappa = \sqrt{2m(V_0 - E)}/\hbar$$

and

$$k = \sqrt{2mE}/\hbar$$



TUNNELING THROUGH A POTENTIAL BARRIER

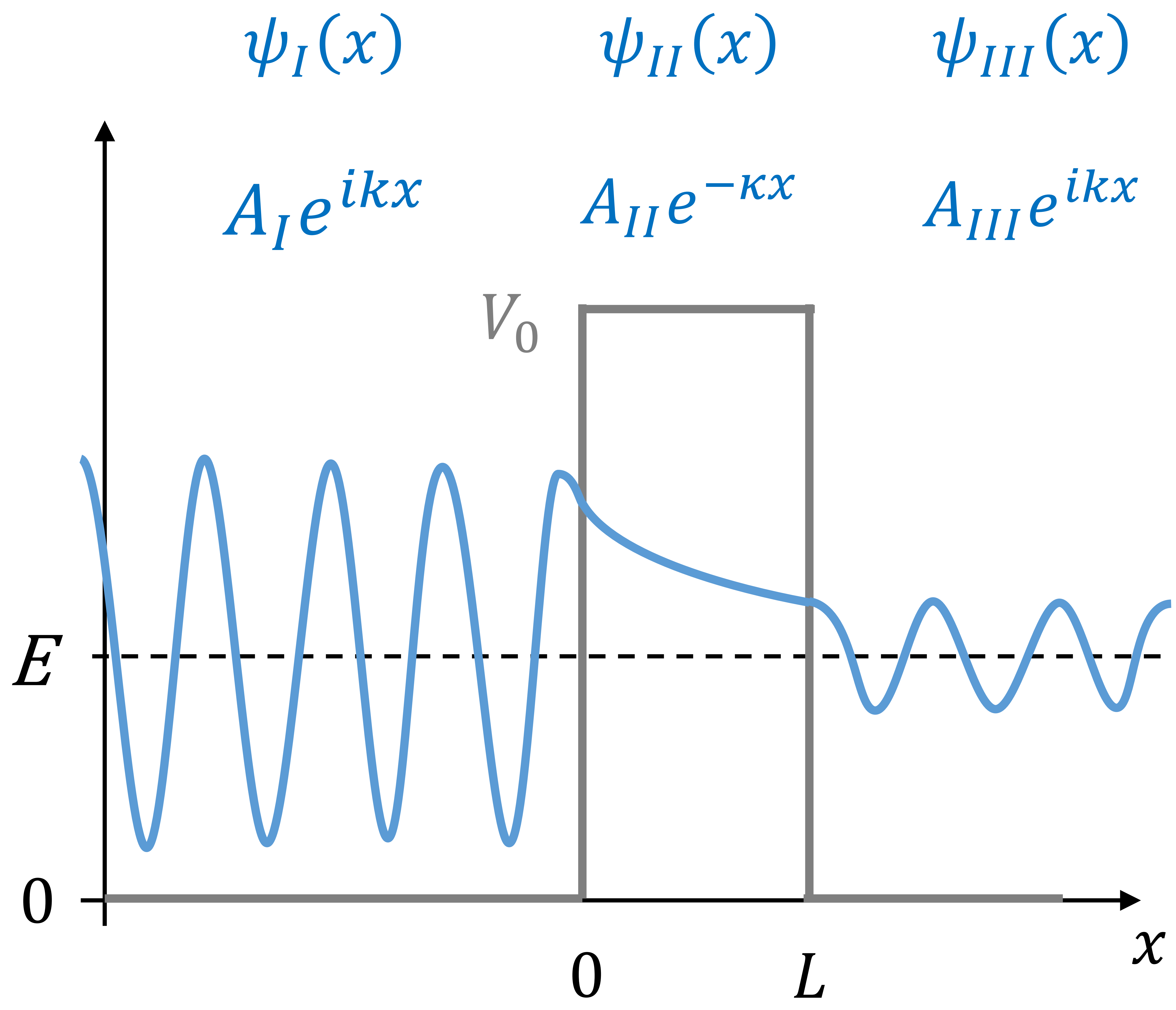
$$\psi_I(x) = A_I e^{ikx}$$
$$\psi_{II}(x) = A_{II} e^{-\kappa x}$$
$$\psi_{III}(x) = A_{III} e^{ikx}$$

- Probability for the wave on the left: $|A_I|^2$
- Similar on the right
- Tunneling probability:

$$T \propto \frac{|A_{III}|^2}{|A_I|^2}$$

- Obtain the amplitudes?

➡ Apply boundary conditions



TUNNELING THROUGH A POTENTIAL BARRIER

$$\psi_I(x) = A_I e^{ikx}$$
$$\psi_{II}(x) = A_{II} e^{-\kappa x}$$
$$\psi_{III}(x) = A_{III} e^{ikx}$$

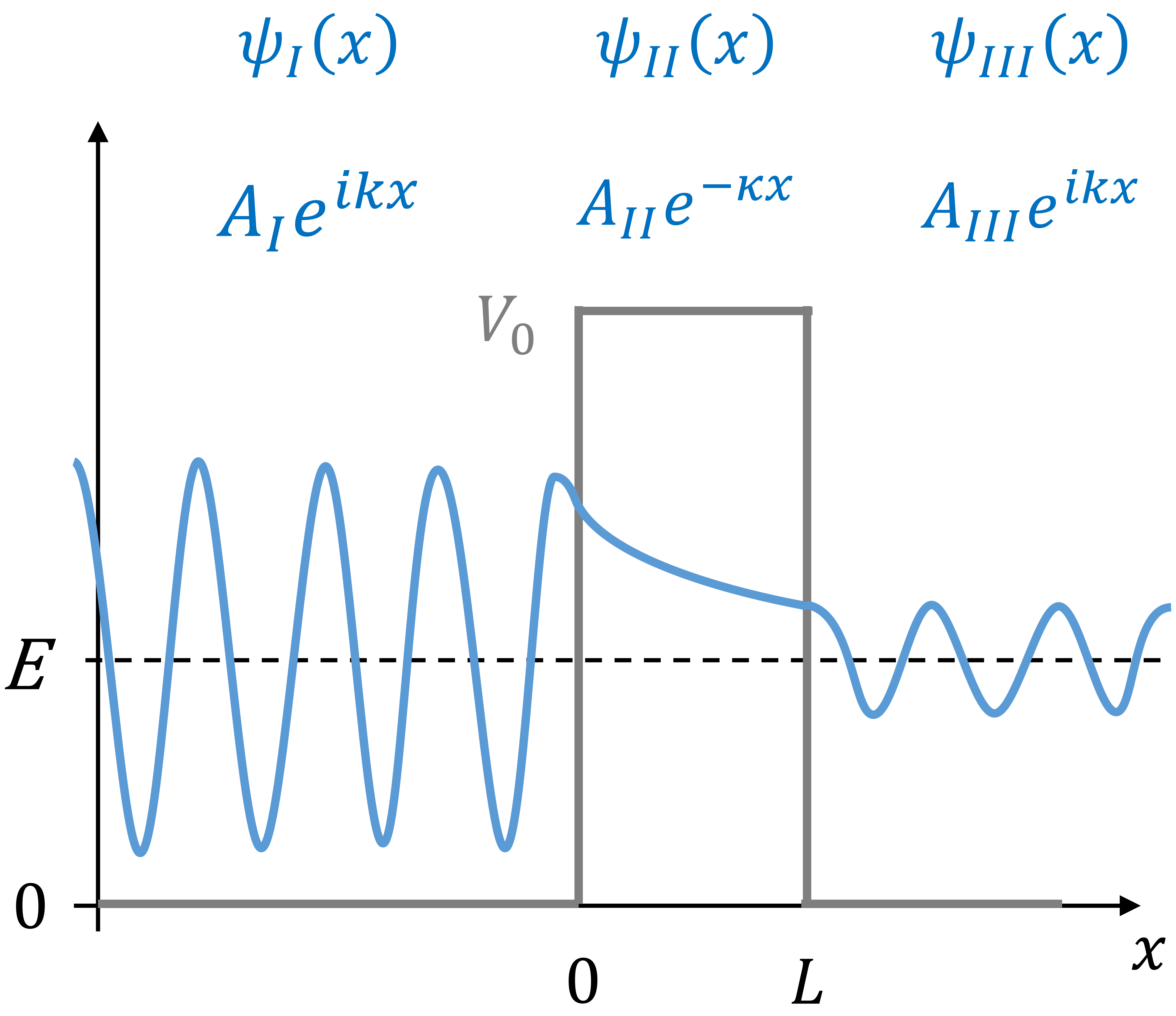
- Boundary conditions:

Continuity of ψ

$$\psi_I(0) = \psi_{II}(0)$$
$$\psi_{II}(L) = \psi_{III}(L)$$

Continuity derivative ψ':

$$\psi'_I(0) = \psi'_{II}(0)$$
$$\psi'_{II}(L) = \psi'_{III}(L)$$



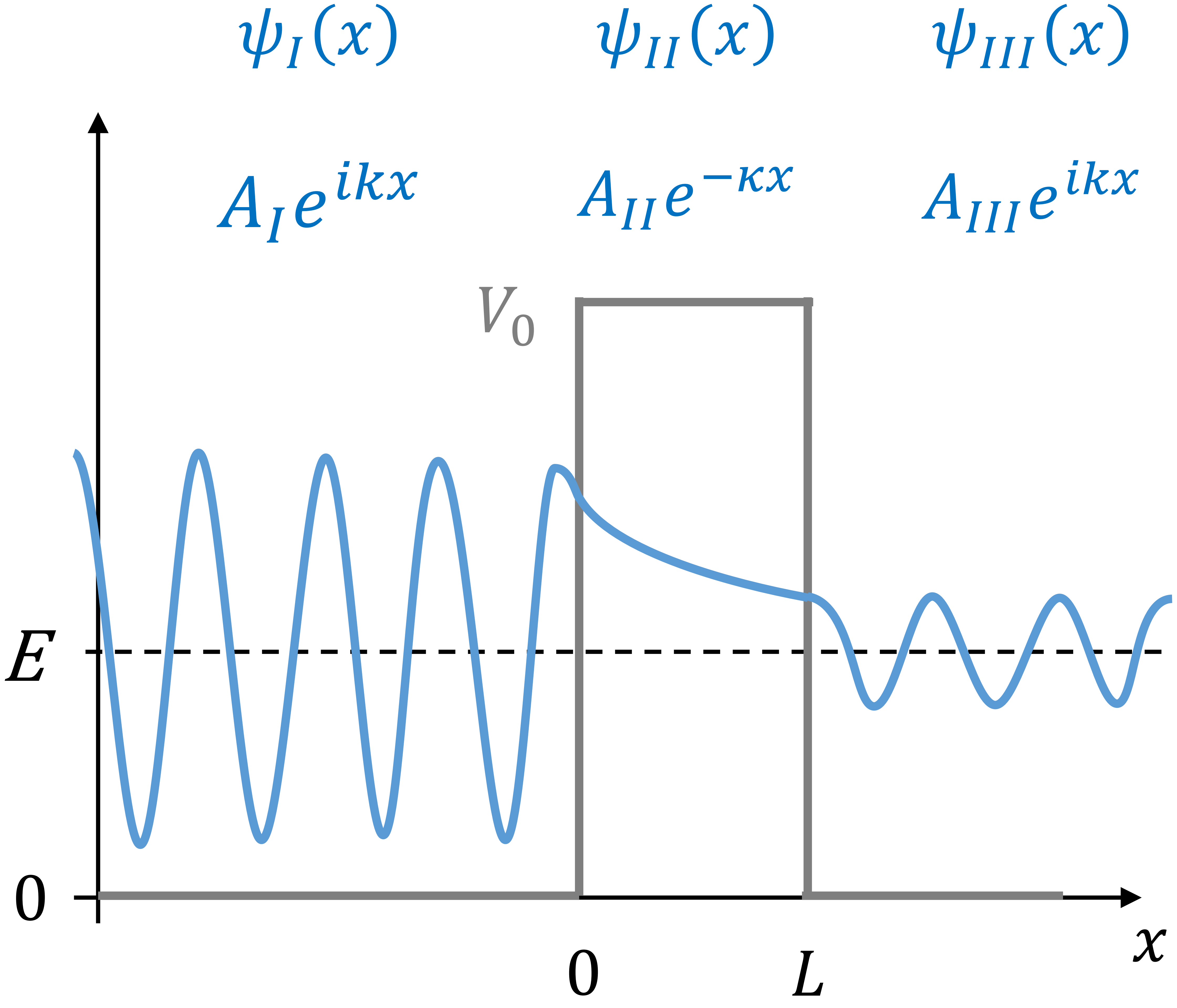
APPROXIMATE TUNNELING PROBABILITY

- For small tunneling probability $T \ll 1$:

$$T \propto \frac{|A_{III}|^2}{|A_I|^2} \approx G e^{-2\kappa L}$$

With

$$\begin{cases} G = 16\dfrac{E}{V_0}\left(1 - \dfrac{E}{V_0}\right) \\ \kappa = \sqrt{2m(V_0 - E)}/\hbar \end{cases}$$



EXAMPLE CALCULATION OF TUNNELING PROBABILITY

Question: Assume an electron energy $E = 2\ \mathrm{eV}$ incident at a barrier with height $V_0 = 5\ \mathrm{eV}$.

Calculate the Tunneling probability T for barrier widths $1\ \mathrm{nm}$ and $0.5\ \mathrm{nm}$

Answer:

$$T \approx Ge^{-2\kappa L} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L}$$

$$\kappa = \sqrt{2m(V_0 - E)}/\hbar$$

EXAMPLE CALCULATION OF TUNNELING PROBABILITY

Question: Assume an electron energy $E = 2 \text{ eV}$ incident at a barrier with height $V_0 = 5 \text{ eV}$.
Calculate the Tunneling probability T for barrier widths 1 nm and 0.5 nm

Answer:

$$G = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) = 16\frac{2}{5}\left(1 - \frac{2}{5}\right) = 3.84$$

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} = \frac{\sqrt{2\ (9.11 \times 10^{-31}\ \text{kg})\ (3 \cdot 1.6 \times 10^{-19}\ \text{J})}}{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}} \approx \frac{\sqrt{90} \times 10^{-25}\ \text{kg}\,\frac{\text{m}}{\text{s}}}{10^{-34}\ kg\,\frac{m^2}{s}} \approx 9 \times 10^{9}\ \text{m}^{-1}$$

For $\text{L} = 1\ \text{nm}$:

$$T \approx Ge^{-2\kappa L} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L} = 3.84\ e^{-18} \approx 6 \times 10^{-8}$$

EXAMPLE CALCULATION OF TUNNELING PROBABILITY

Question: Assume an electron energy $E = 2\ \text{eV}$ incident at a barrier with height $V_0 = 5\ \text{eV}$.
Calculate the Tunneling probability T for barrier widths $1\ \text{nm}$ and $0.5\ \text{nm}$

Answer:

$$G = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) = 16\frac{2}{5}\left(1 - \frac{2}{5}\right) = 3.84$$

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For $\text{L} = 1\ \text{nm}$:

$$T \approx Ge^{-2\kappa L} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L} = 3.84\ e^{-18} \approx 7 \times 10^{-8}$$

For $\text{L} = 0.5\ \text{nm}$:

$$T \approx Ge^{-2\kappa L} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L} = 3.84\ e^{-9} \approx 5 \times 10^{-4}$$

EXAMPLE CALCULATION OF TUNNELING PROBABILITY

Question: Assume an electron energy $E = 2\ \mathrm{eV}$ incident at a barrier with height $V_0 = 5\ \mathrm{eV}$.
Calculate the Tunneling probability T for barrier widths $1\ \mathrm{nm}$ and $0.5\ \mathrm{nm}$

Answer:

$$G = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) = 16\frac{2}{5}\left(1 - \frac{2}{5}\right) = 3.84$$

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} = \frac{\sqrt{2\ (9.11 \times 10^{-31}\ \mathrm{kg})\ (3 \cdot 1.6 \times 10^{-19}\ \mathrm{J})}}{1.055 \times 10^{-34}\ \mathrm{J \cdot s}} \approx \frac{\sqrt{90} \times 10^{-25}\ \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}}}{10^{-34}\ kg\frac{m^2}{s}} \approx 9 \times 10^{9}\ \mathrm{m}^{-1}$$

For $\mathrm{L} = 1\ \mathrm{nm}$:

$$T \approx Ge^{-2\kappa L} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L} = 3.84\ e^{-18} \approx 7 \times 10^{-8}$$

For $\mathrm{L} = 0.5\ \mathrm{nm}$:

$$T \approx Ge^{-2\kappa L} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L} = 3.84\ e^{-9} \approx 5 \times 10^{-4}$$

Very sensitive to the width of the barrier

For larger macroscopic mass tunneling becomes very small

APPLICATION OF TUNNELING PROBABILITY

- Resonant tunneling diode
- **Combines** the **Finite well bound states** and **tunneling** through a barrier

SUMMARY TUNNELING

- Quantum particles can tunnel through potential barriers
- Transmission + reflection = 1
- Tunneling probability for low T values can be approximated

$$T \approx Ge^{-2\kappa L} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L}$$

- Tunneling probability is sensitive to barrier width
- Macroscopic objects have very small tunneling probability