PHOT 222: Quantum Photonics
LECTURE 01

Michaël Barbier, Spring semester (2024-2025)

OVERVIEW OF THE COURSE

week	topic	Serway	Young
Week 1	**Relativity**	**Ch. 39**	**Ch. 37**
Week 2	Waves and Particles		
Week 3	Wave packets and Uncertainty		
Week 4	The Schrödinger equation and Probability		
Week 5	**Midterm exam 1**		
Week 6	Quantum particles in a potential		
Week 7	Harmonic oscillator		
Week 8	Tunneling through a potential barrier		
Week 9	The hydrogen atom, absorption/emission spectra		
Week 10	**Midterm exam 2**		
Week 11	Many-electron atoms		
Week 12	Pauli-exclusion principle		
Week 13	Atomic bonds and molecules		
Week 14	Crystalline materials and energy band structure		

PROBLEM BETWEEN NEWTON’S & MAXWELL’S EQUATIONS

- Newtonian mechanics: Galilean relativity
- Maxwell’s equations: A **constant speed of light** was found
- Experimentally we cannot accelerate electrons beyond the speed of light

NEWTON'S MECHANICS: GALILEAN RELATIVITY

- **Galilean relativity:**

Laws of mechanics same in all inertial reference frames

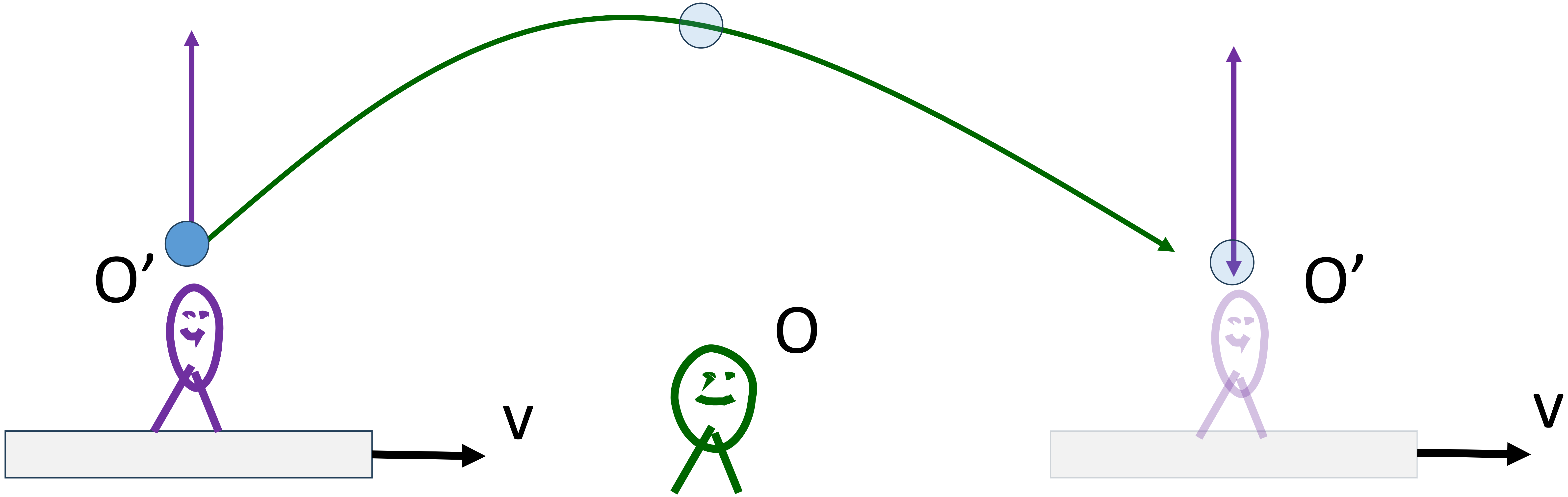
- **Inertial frame** of reference: object with no force acting on it does not accelerate
- **Any frame moving with constant velocity with respect to an inertial frame** is also an inertial frame
- No absolute inertial frame, but time is an absolute parameter

NEWTON’S MECHANICS: GALILEAN RELATIVITY

Laws of mechanics same in all inertial reference frames

Inertial frames O and O’

O standing still, O’ on a moving platform



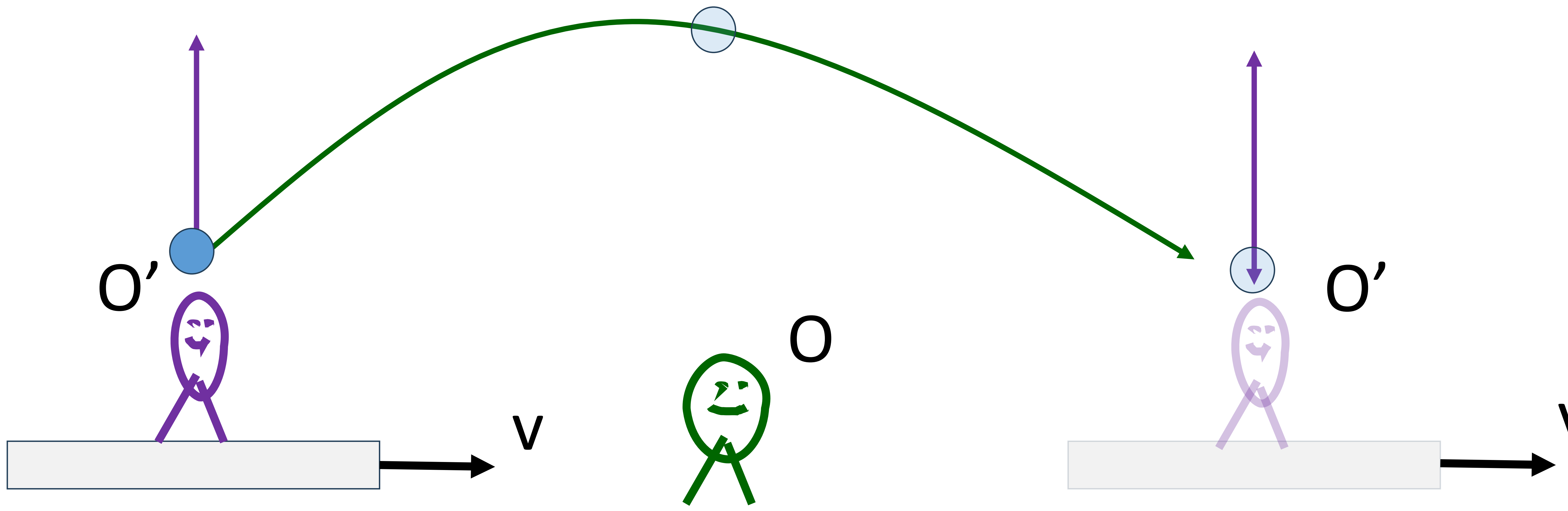
NEWTON'S MECHANICS: GALILEAN RELATIVITY

Laws of mechanics same in all inertial reference frames

Inertial frames O and O'

O standing still, O' on a moving platform

O: ball parabolic trajectory O': ball goes up and down

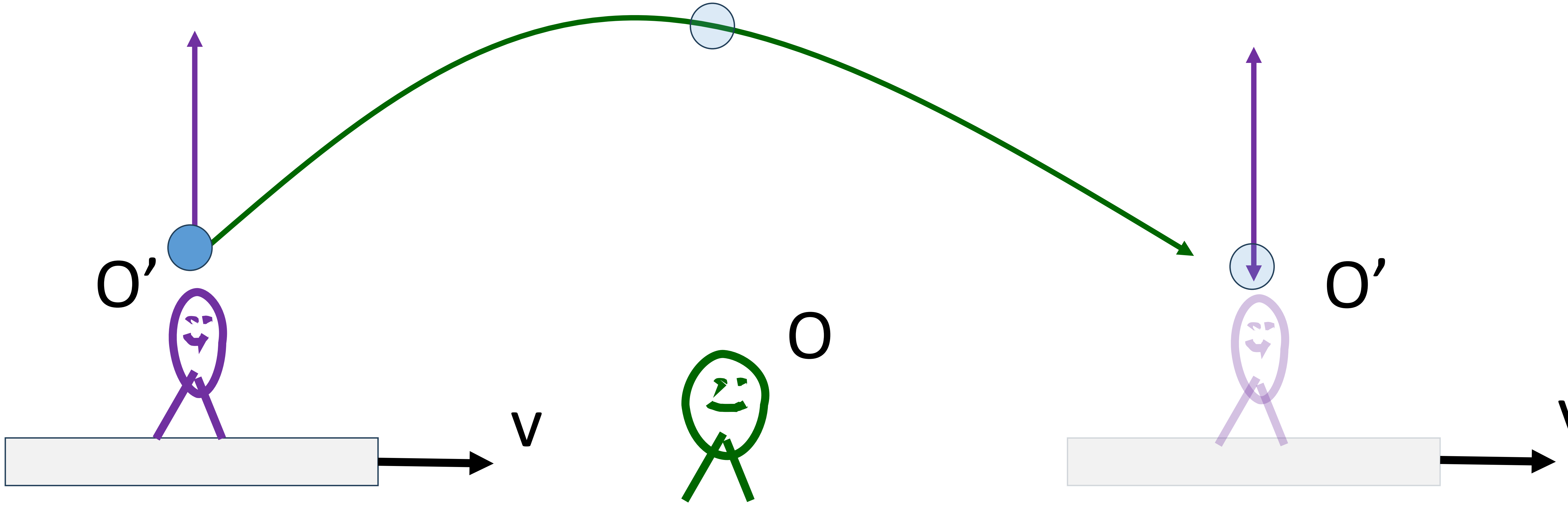


NEWTON'S MECHANICS: GALILEAN RELATIVITY

Laws of mechanics same in all inertial reference frames

Galilean space-time transformation:

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$



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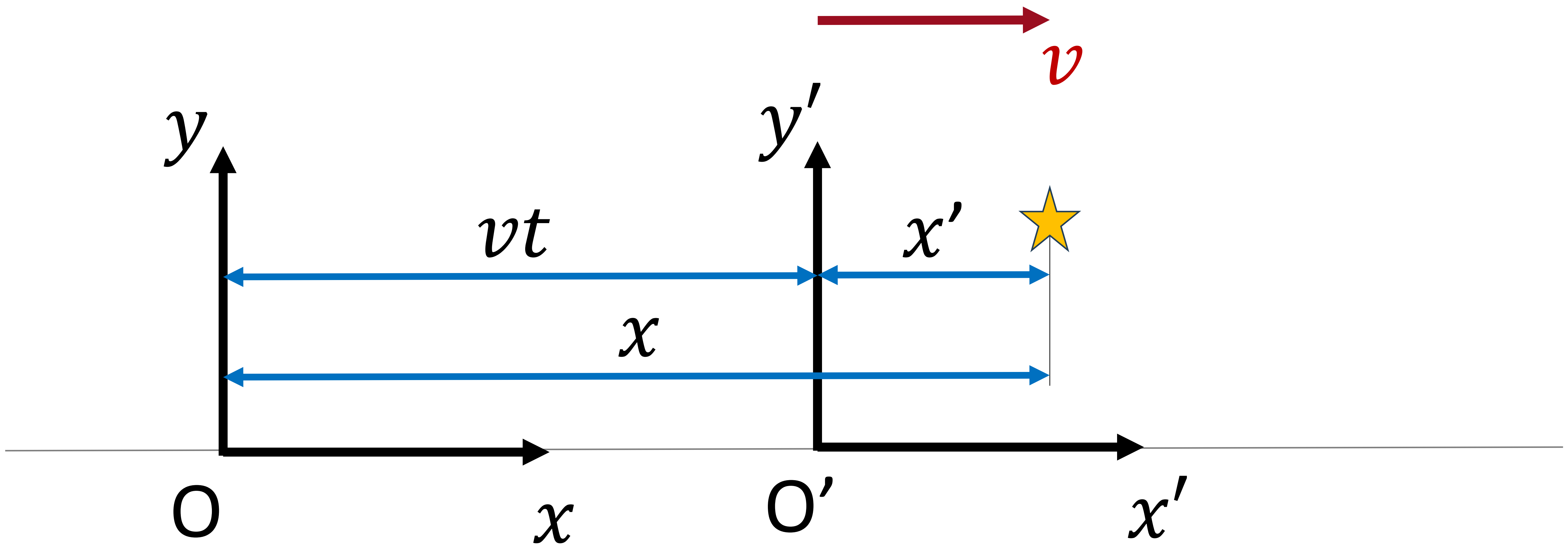


NEWTON'S MECHANICS: GALILEAN RELATIVITY

Laws of mechanics same in all inertial reference frames

Galilean space-time transformation:

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

Galilean velocity transformation:

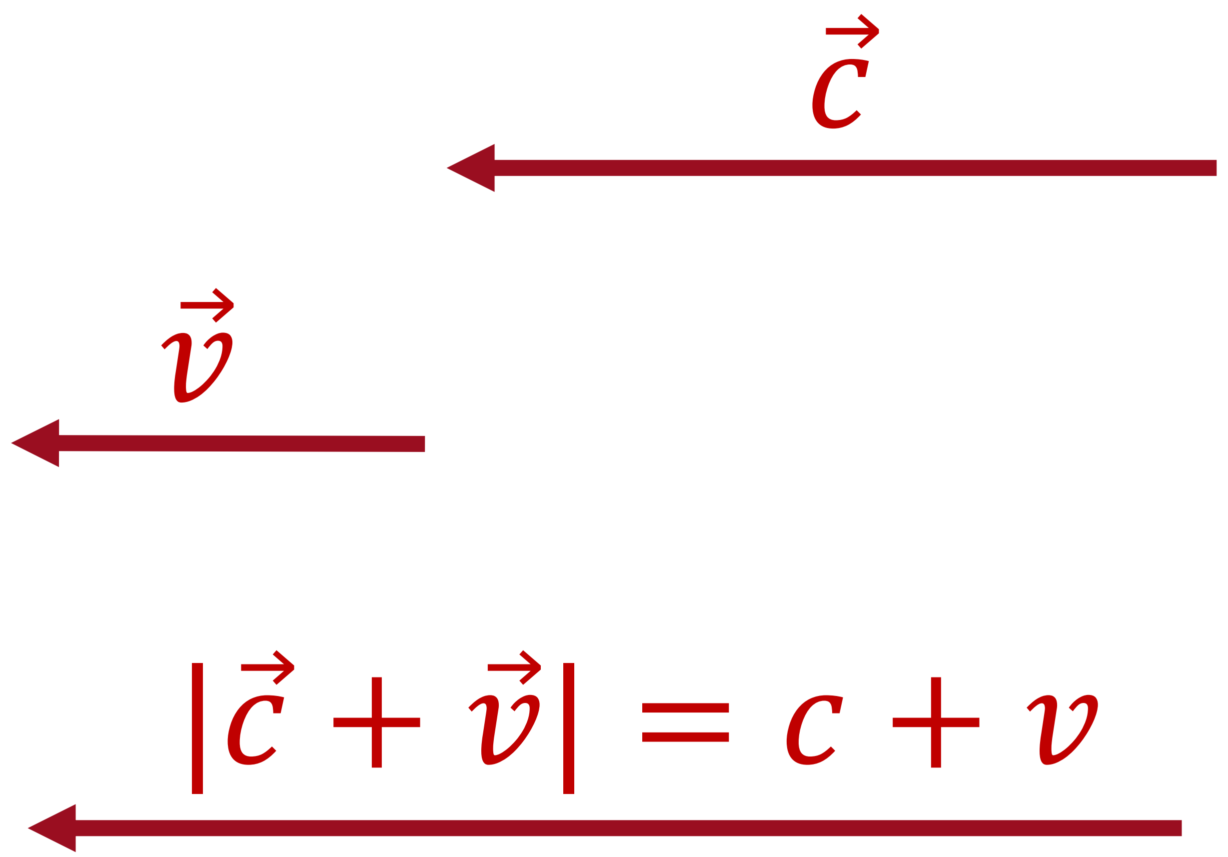
$$\boldsymbol{u_x' = u_x - v}, \quad u_y' = u_y, \quad u_z' = u_z, \quad \boldsymbol{t' = t}$$

PROBLEM BETWEEN NEWTON'S & MAXWELL'S EQUATIONS

- Newtonian mechanics: Galilean relativity $\implies$ $\boldsymbol{u'_x = u_x - v}$

- Maxwell's equations: A **constant speed of light** was found

Light moving along frame



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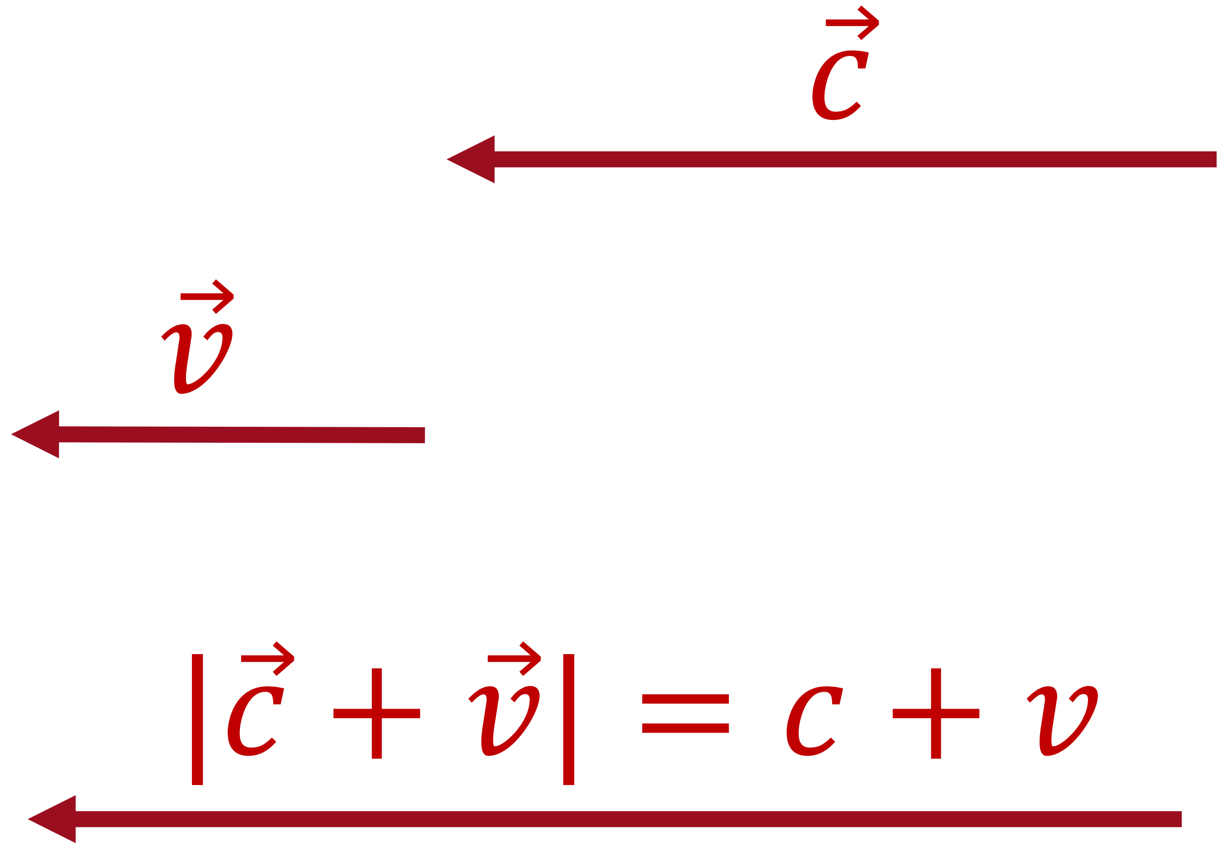
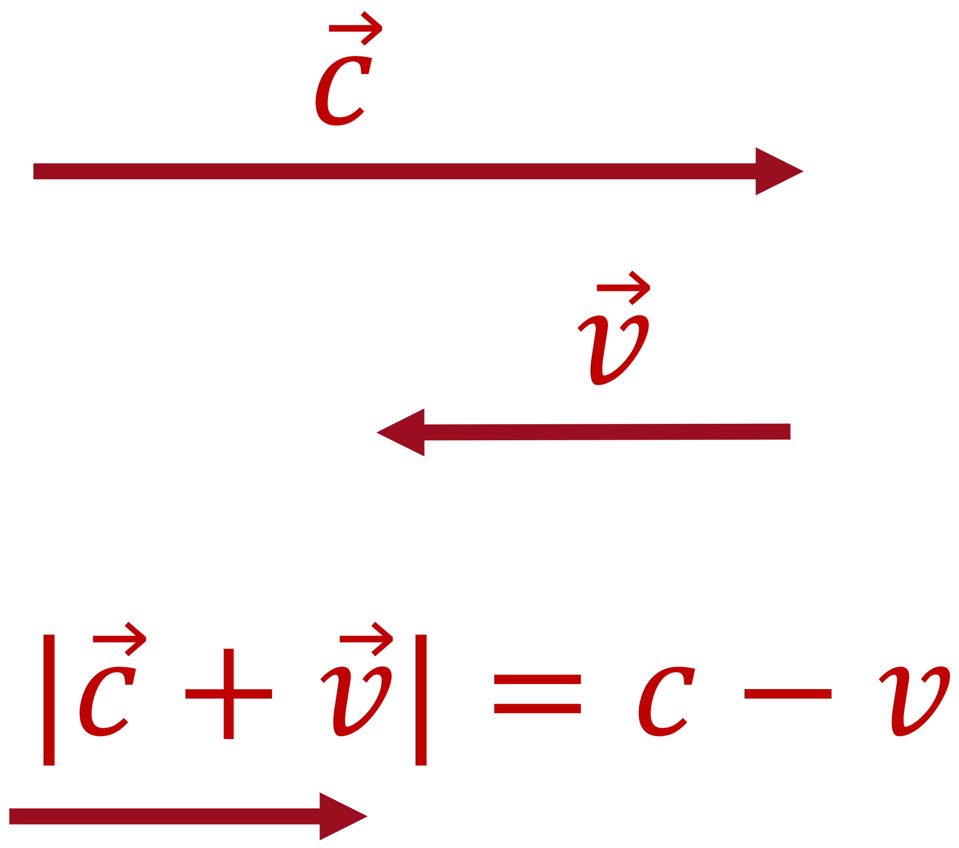


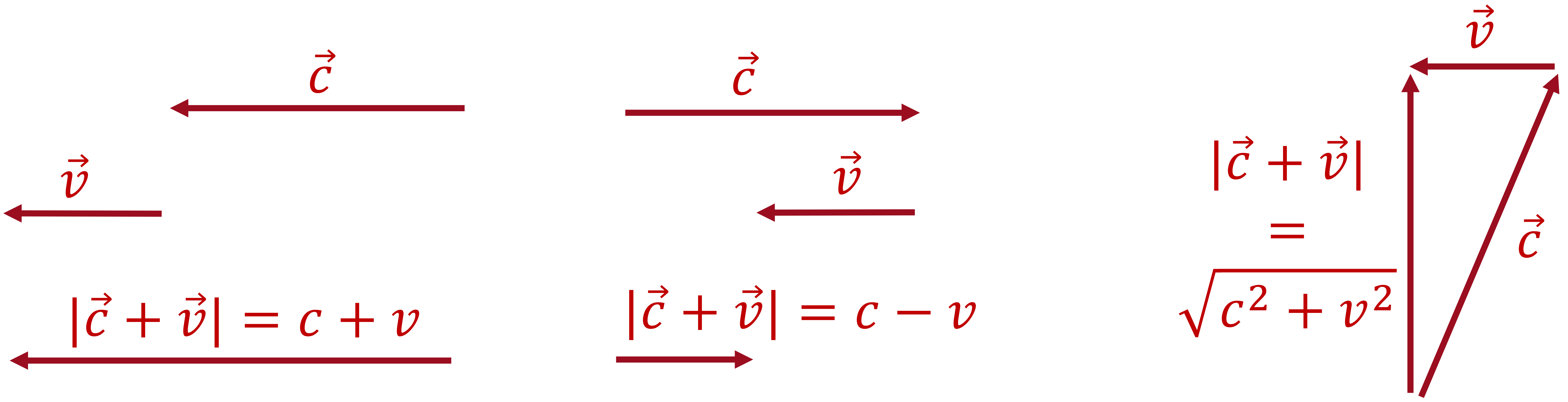
Light moving against frame



PROBLEM BETWEEN NEWTON’S & MAXWELL’S EQUATIONS

- Newtonian mechanics: Galilean relativity $\Longrightarrow$ $\boldsymbol{u'_x = u_x - v}$
- Maxwell’s equations: A **constant speed of light** was found

along frame **against frame** **perpendicular**



PROBLEM BETWEEN NEWTON'S & MAXWELL'S EQUATIONS

- Newtonian mechanics: Galilean relativity

  Light speed depends on frame: $c - v \neq c + v$

- Maxwell's equations: A **constant speed of light $\boldsymbol{c}$** was found

SOLUTION: LIGHT WAVES MOVE IN A MEDIUM: “ETHER”

Idea of the Ether:

- The Ether would be what water is for waves in the sea
- We cannot see the Ether, but it fills space
- Electromagnetic waves have a fixed inertial frame
- We could outrun light like water waves or sound waves

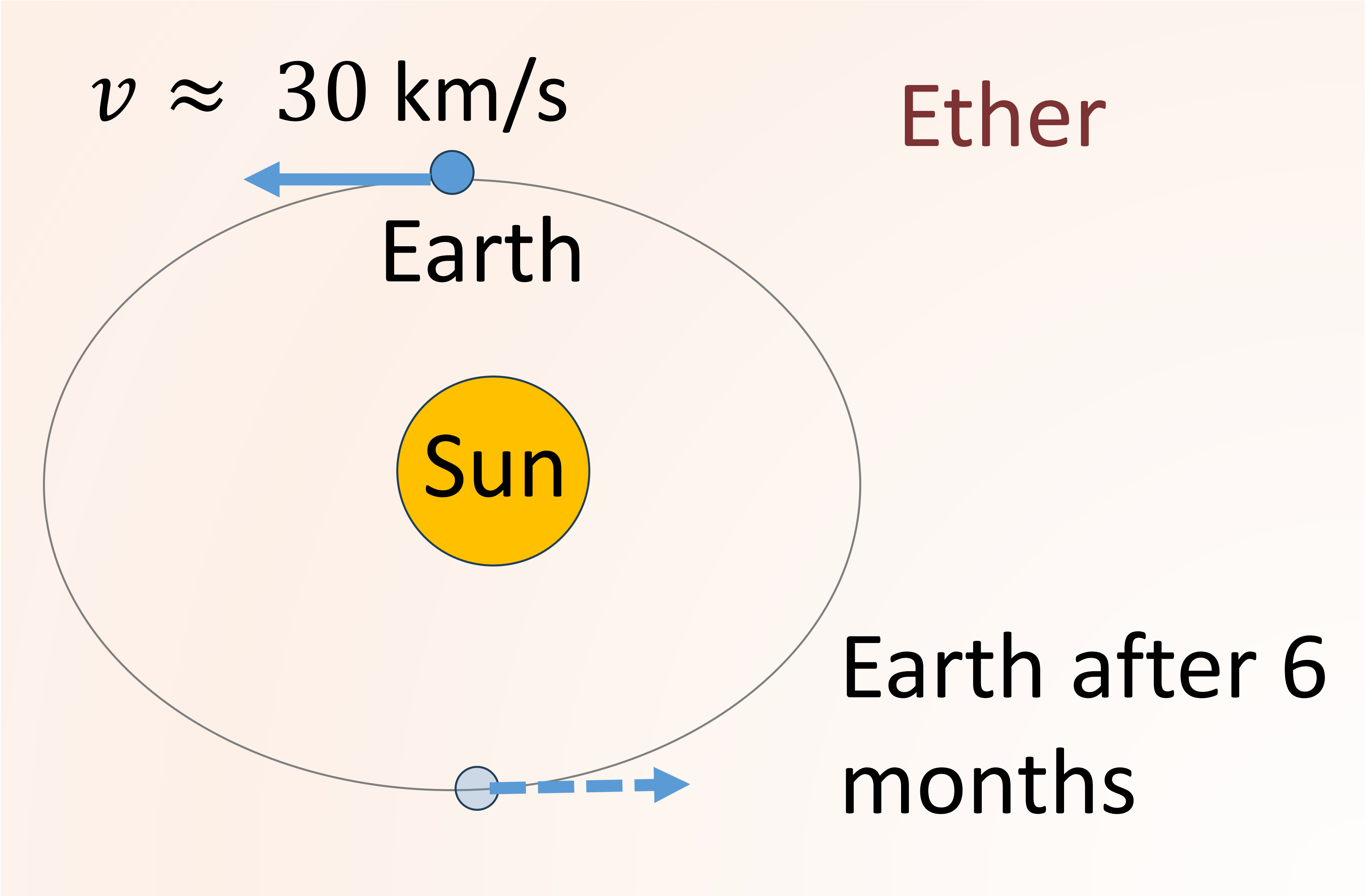
Newton’s equations and Galilean relativity still valid!

(Attempted) Proof of the Ether:

Measure the velocity of the Ether:
Michelson-Morley experiment

THE MICHELSON-MORLEY EXPERIMENT

- Measuring the Ether wind: our velocity compared to the Ether
- Assume that the Ether does not move along with the Earth
- Earth rotates around the Sun with velocity: $v \approx 30$ km/s
- Light has velocity of $c = 3 \times 10^5$ km/s $\approx 10^4\, v$



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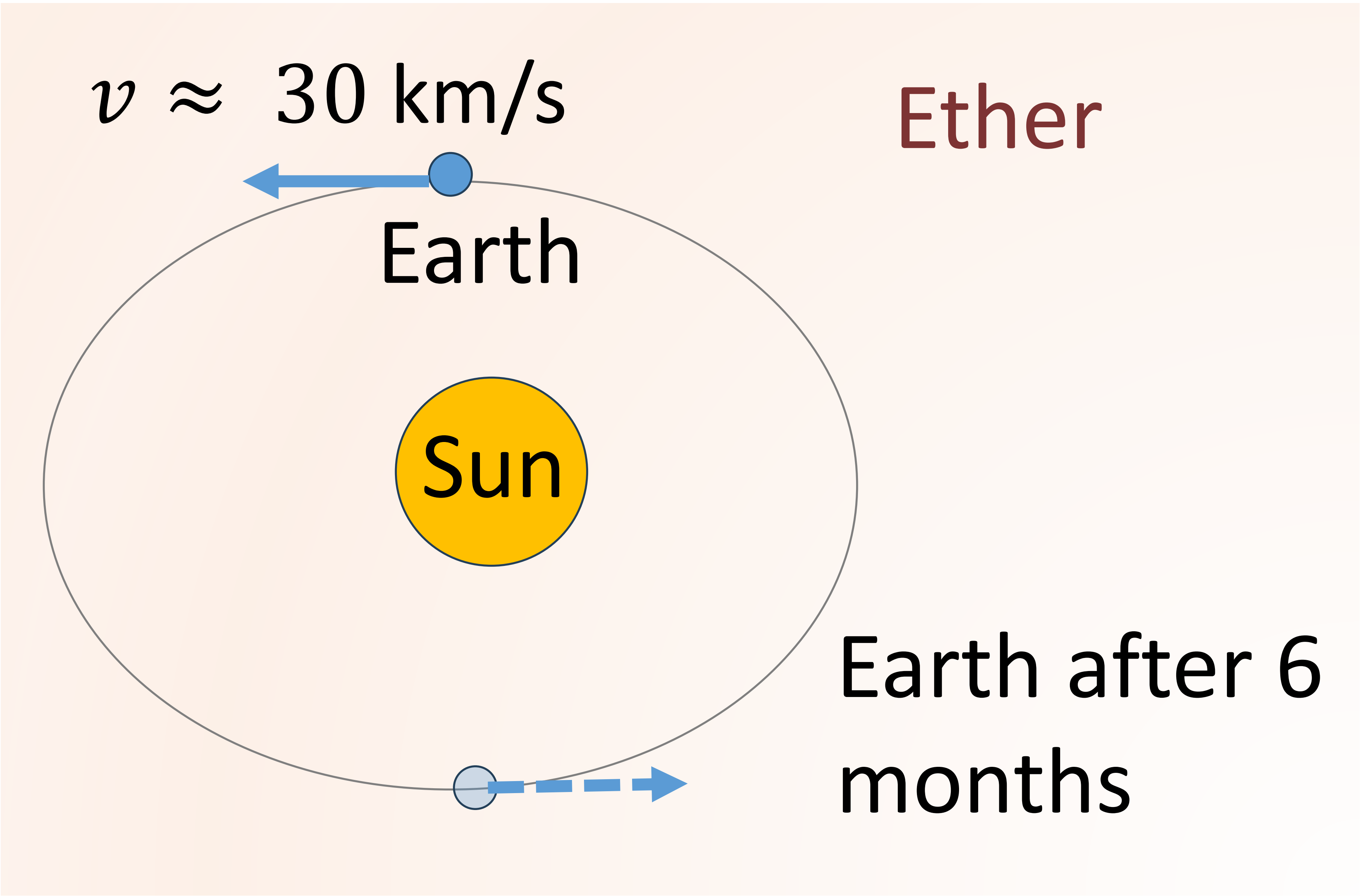
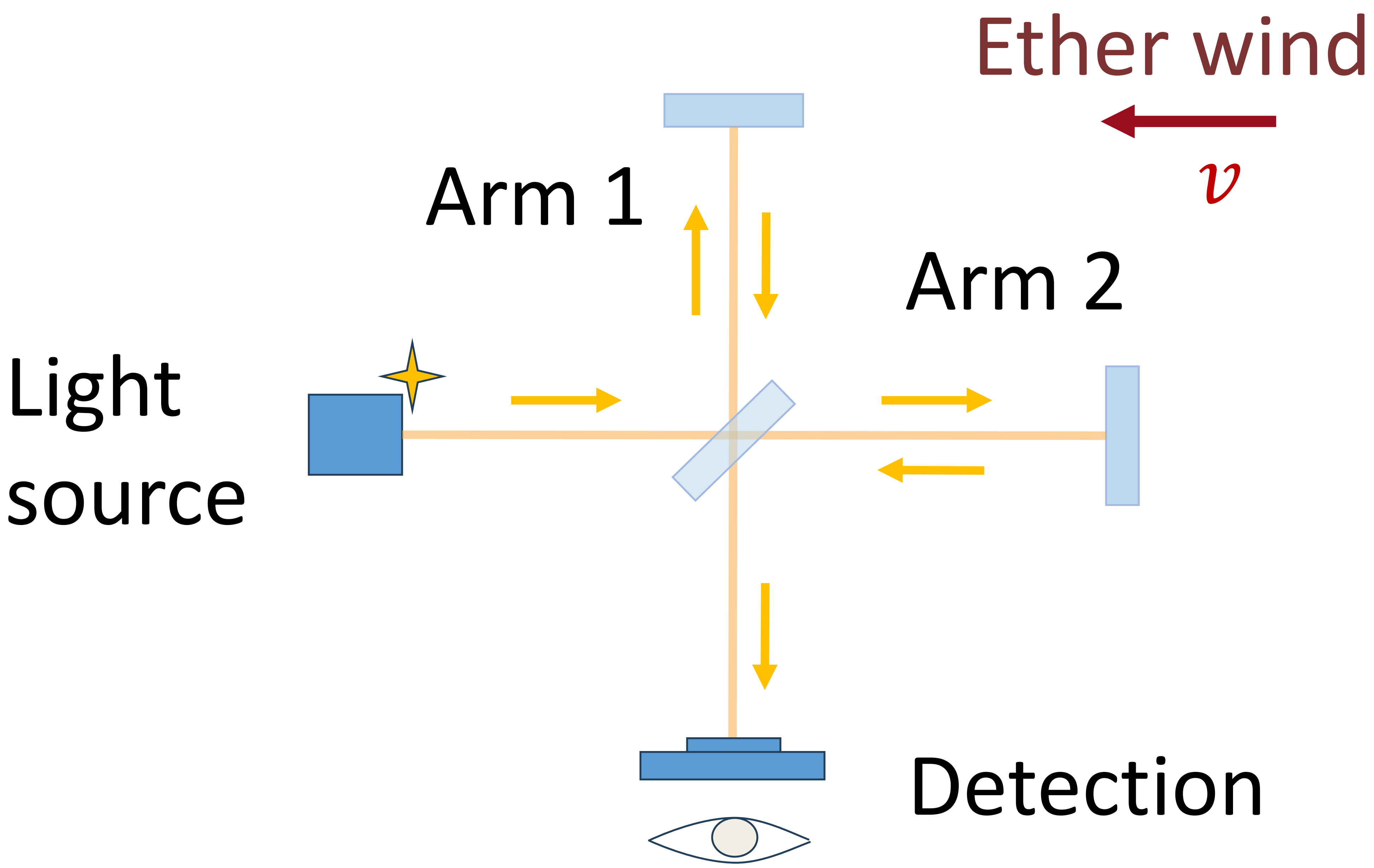
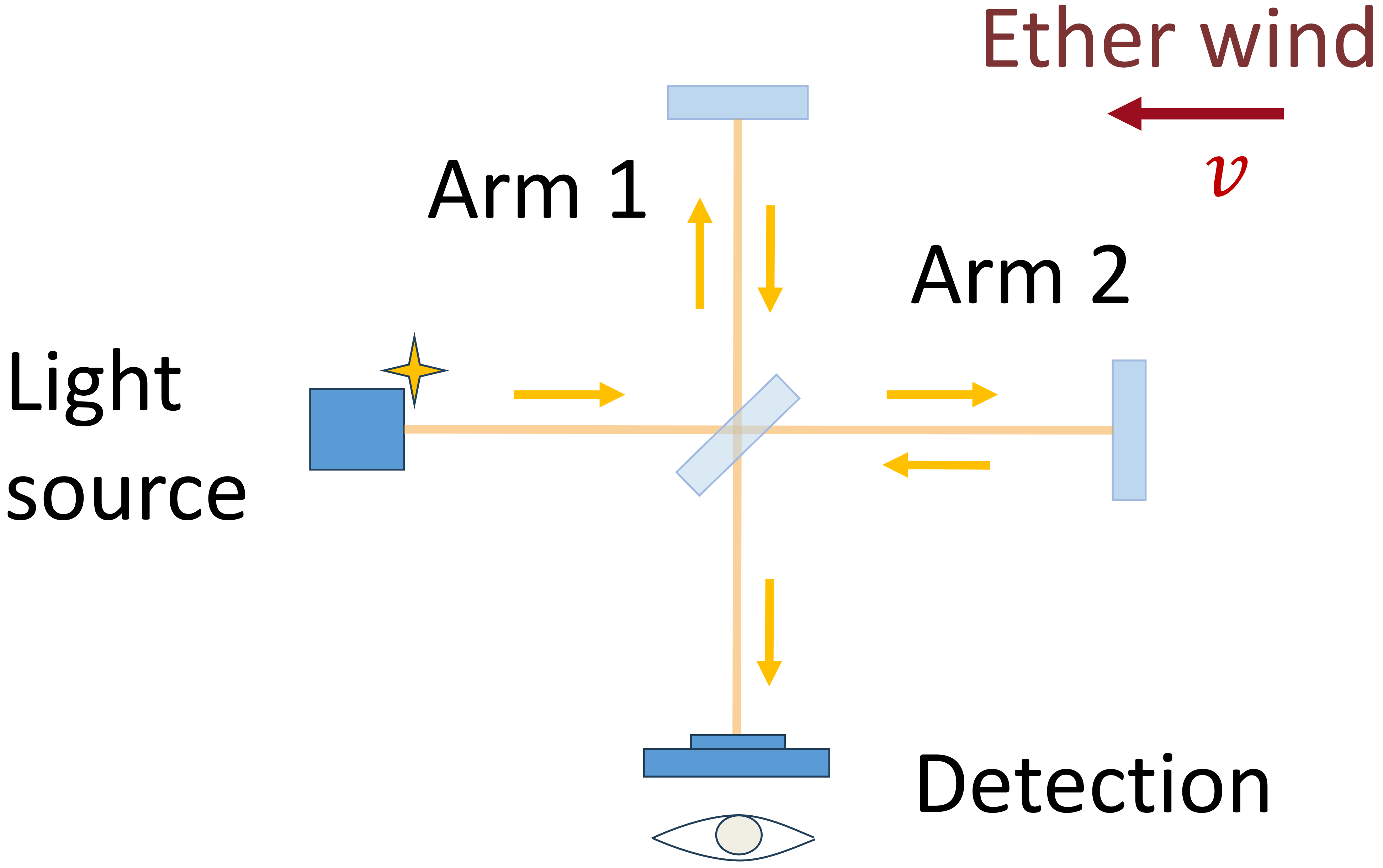
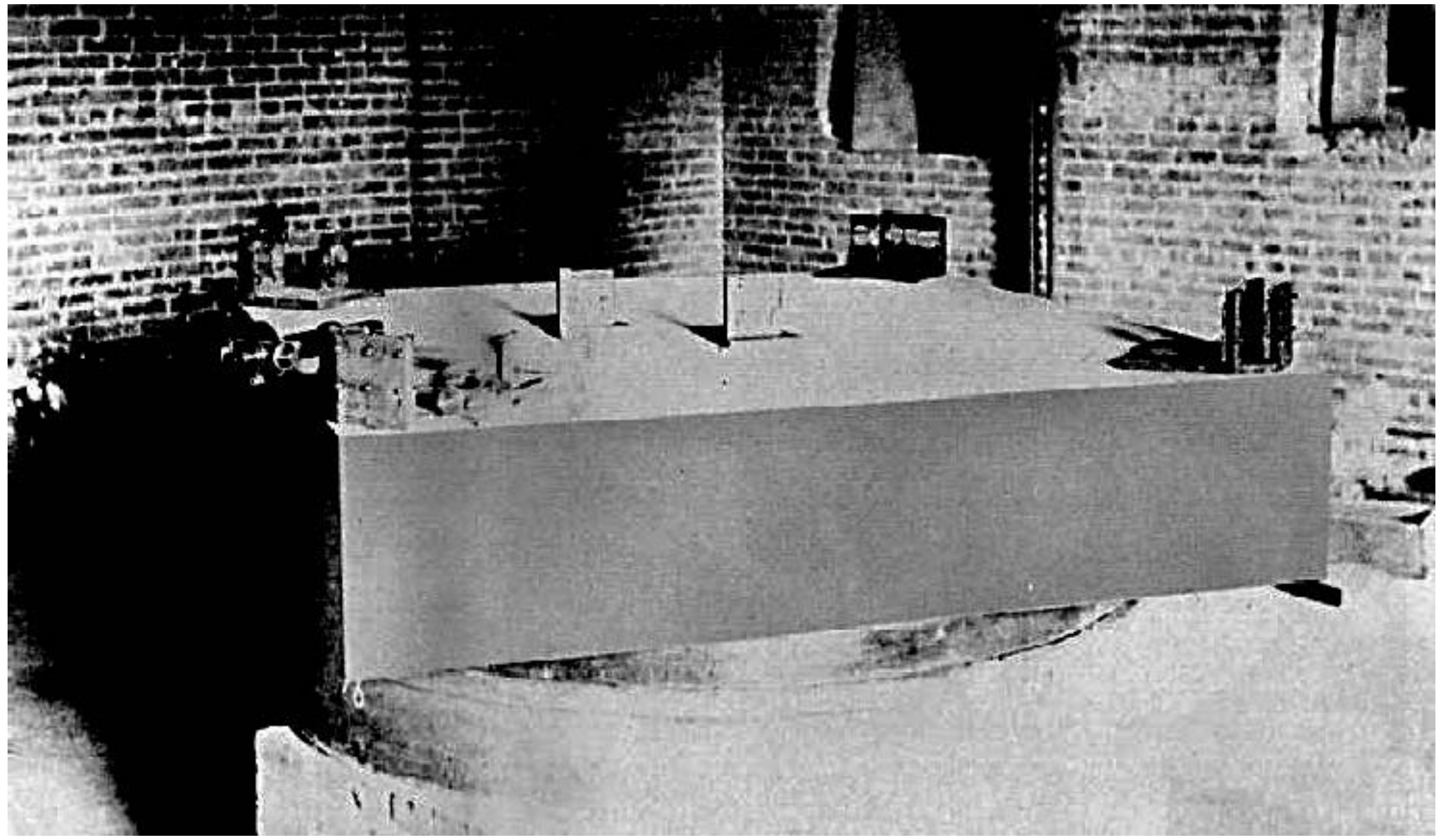


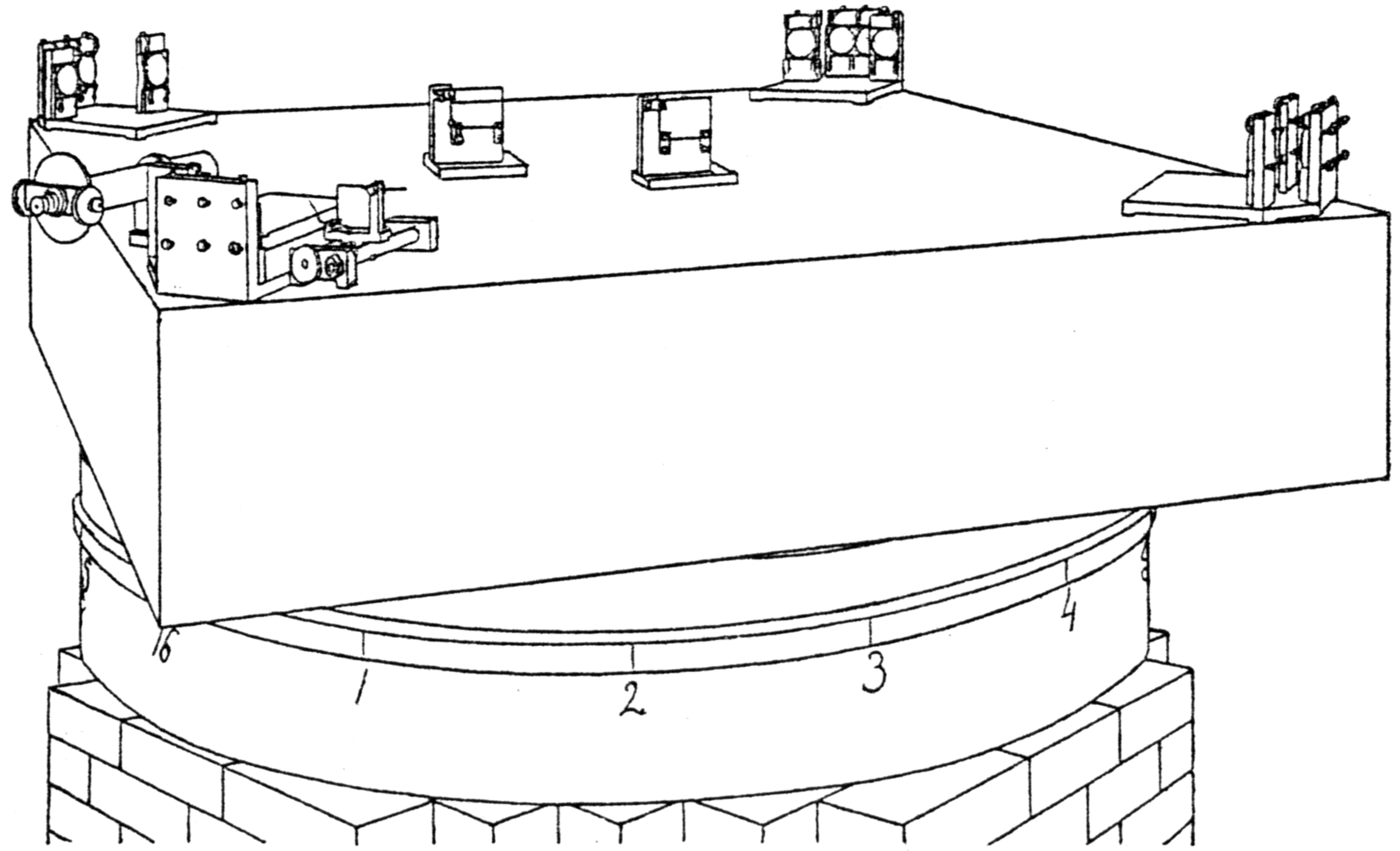


THE MICHELSON-MORLEY EXPERIMENT

- Turn interferometer parallel vs. perpendicular to Ether wind

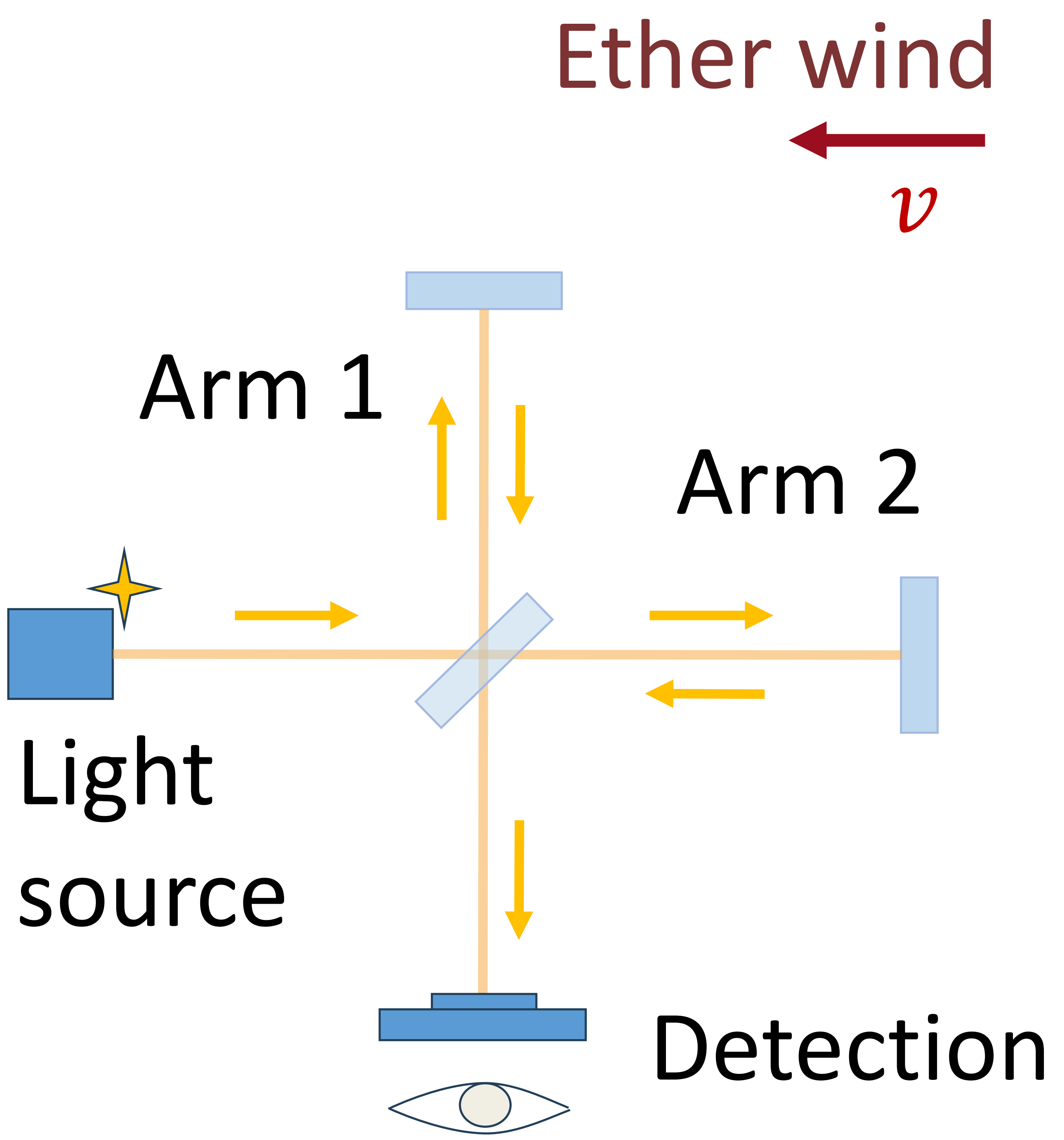






THE MICHELSON-MORLEY EXPERIMENT

Calculate time for pulse to travel along different arms 1 & 2



- Arm 2: along Ether wind

$$\Delta t_2 = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

- Arm 1: perpendicular to the Ether wind

$$\Delta t_1 = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

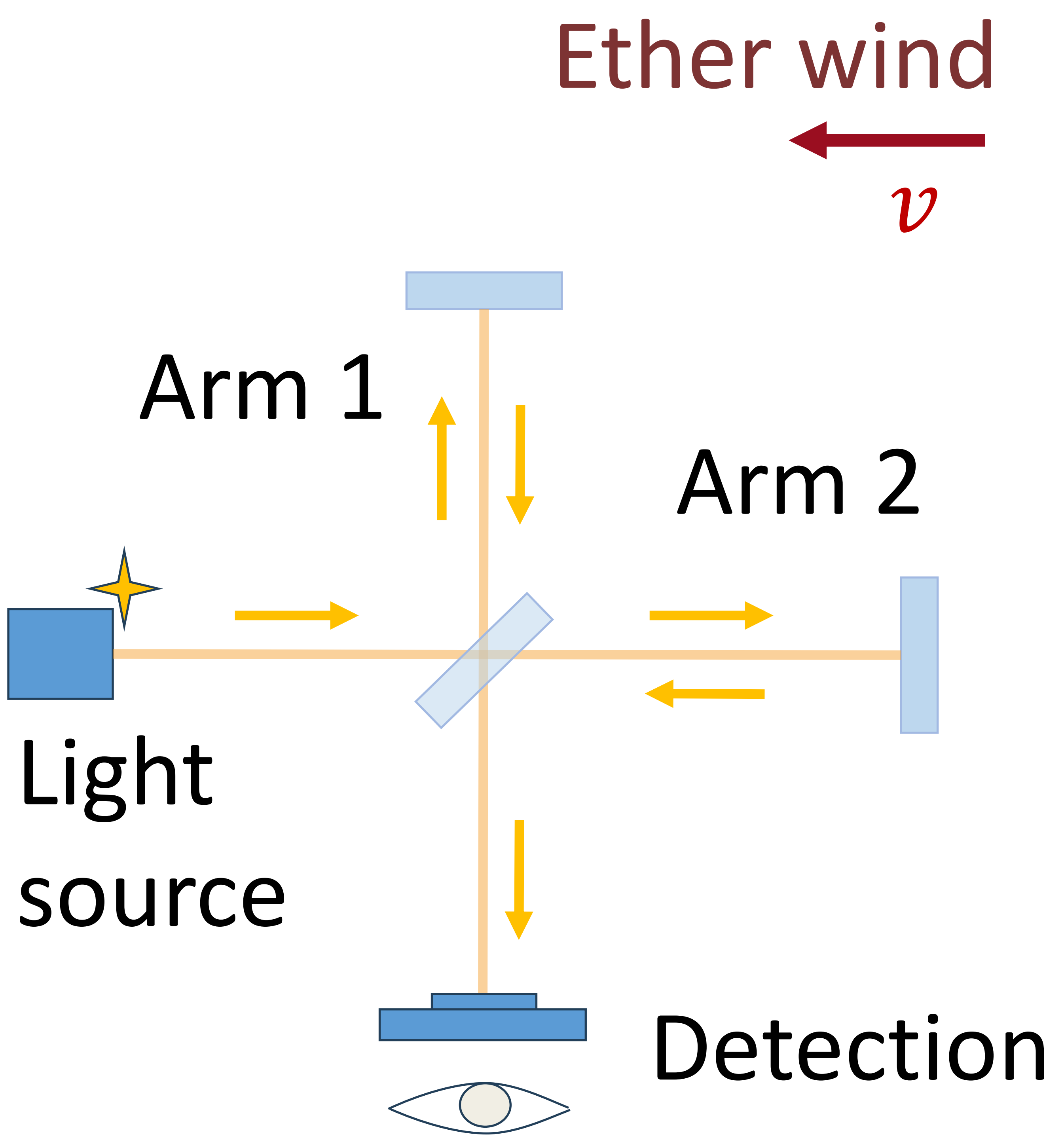
THE MICHELSON-MORLEY EXPERIMENT

Calculate time for pulse to travel along different arms 1 & 2

$$\Delta \mathrm{t} = \Delta t_2 - \Delta t_1$$

$$= \frac{2L}{c}\left[\left(1-\frac{v^2}{c^2}\right)^{-1} - \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right]$$

$$\approx \frac{Lv^2}{c^3}$$

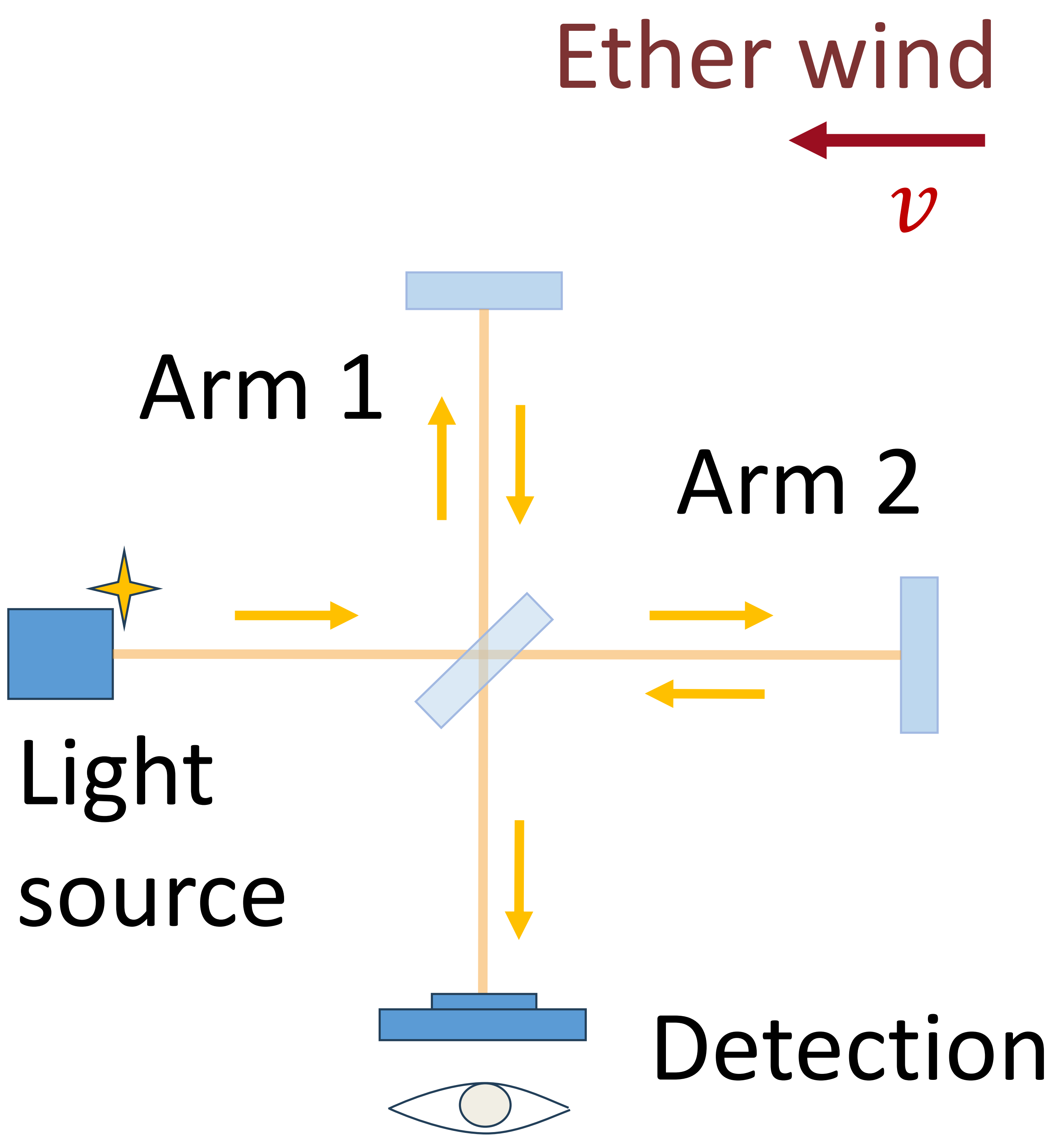


THE MICHELSON-MORLEY EXPERIMENT

Calculate time for pulse to travel along different arms 1 & 2

$$\Delta t = \Delta t_2 - \Delta t_1 \approx \frac{Lv^2}{c^3}$$

- Different time $\Longrightarrow$ phase difference
- Turn interferometer to swap arms 1 & 2



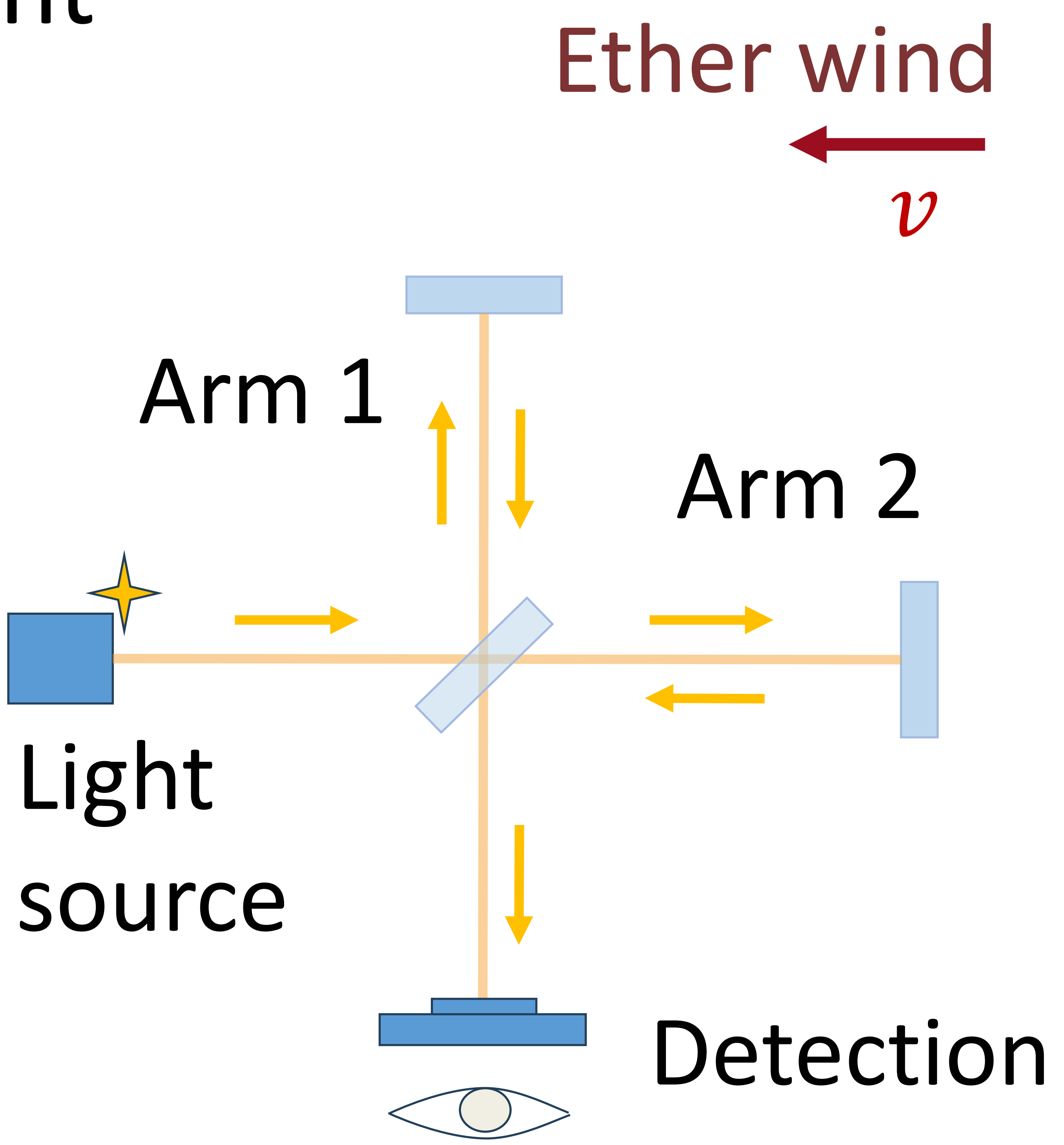
THE MICHELSON-MORLEY EXPERIMENT

Calculate time for pulse to travel along different arms 1 & 2

$$\Delta t = \Delta t_2 - \Delta t_1 \approx \frac{Lv^2}{c^3}$$

- Different time $\Longrightarrow$ phase difference
- Turn interferometer to swap arms 1 & 2
- Time difference is twice Δt, distance:

$$\Delta d = c\, 2\Delta t \approx \frac{2Lv^2}{c^2} \Rightarrow \Delta\phi = \frac{2Lv^2}{\lambda\, c^2} \approx 0.44$$



THE MICHELSON-MORLEY EXPERIMENT

- For moving Earth a large phase difference should be detected

$$\Delta \mathrm{d} \; = \mathrm{c}\; 2\Delta t \approx \frac{2Lv^2}{c^2} \;\; \Rightarrow \;\; \Delta\phi = \frac{2Lv^2}{\lambda\, c^2} \approx 0.44$$

- But $\Delta\phi = 0$ according to the experiment

- No Ether wind could be detected, therefore the assumption of the existence of an Ether is wrong

SUMMARY HISTORICAL ARGUMENTS

- Maxwell's law result in a constant speed of light
- Galilean relativity (Newton's mechanics) not compatible with constant speed of light
- Michelson-Morley experiment: use Earth's velocity to measure Ether wind
- Ether: the medium of light does not exist (proven by the Michelson-Morley experiments)

Postulates of the special theory of relativity

Principle of relativity: All laws of physics must be valid in all inertial frames

Constant speed of light in vacuum $c = 3 \times 10^{8}$ m/s

Postulates of the special theory of relativity

Principle of relativity: All laws of physics (**including electromagnetism and the constant speed of light**) must be valid in all inertial frames

Constant speed of light in vacuum $c = 3 \times 10^{8}$ m/s

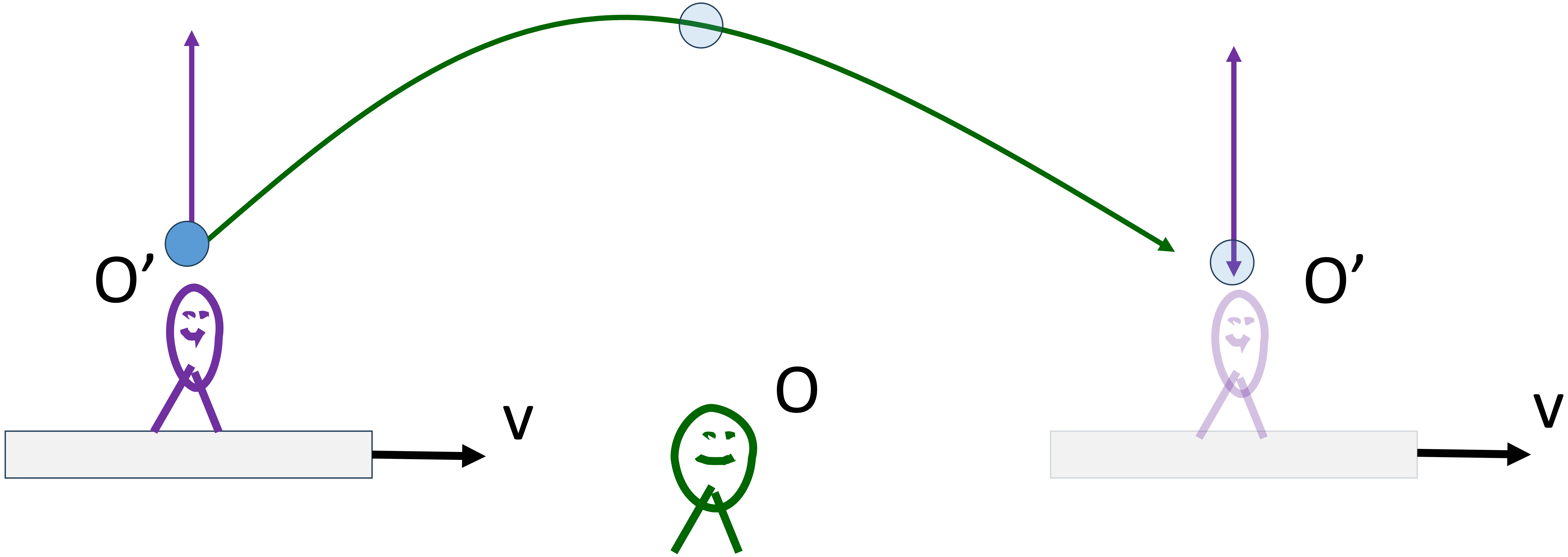
SPECIAL RELATIVITY

- Differences with Galilean relativity: an inertial frame cannot go faster than light.
- Galilean transformation not correct: Lorentz transformation
- This will break the concepts of:
  - Simultaneous events or simultaneity
  - Time intervals
  - Time as a universal parameter
  - Distance

Different observers will have different ideas about “reality”

SPECIAL RELATIVITY: RELATIVITY OF TIME

- Classically time is the same for all observers:
  - it is a universal parameter.
  - If one puts a clock in a car, it still gives the same time.
- Galilean transformations:
  - do not alter time, only position, velocity, etc.



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- Galilean transformations:
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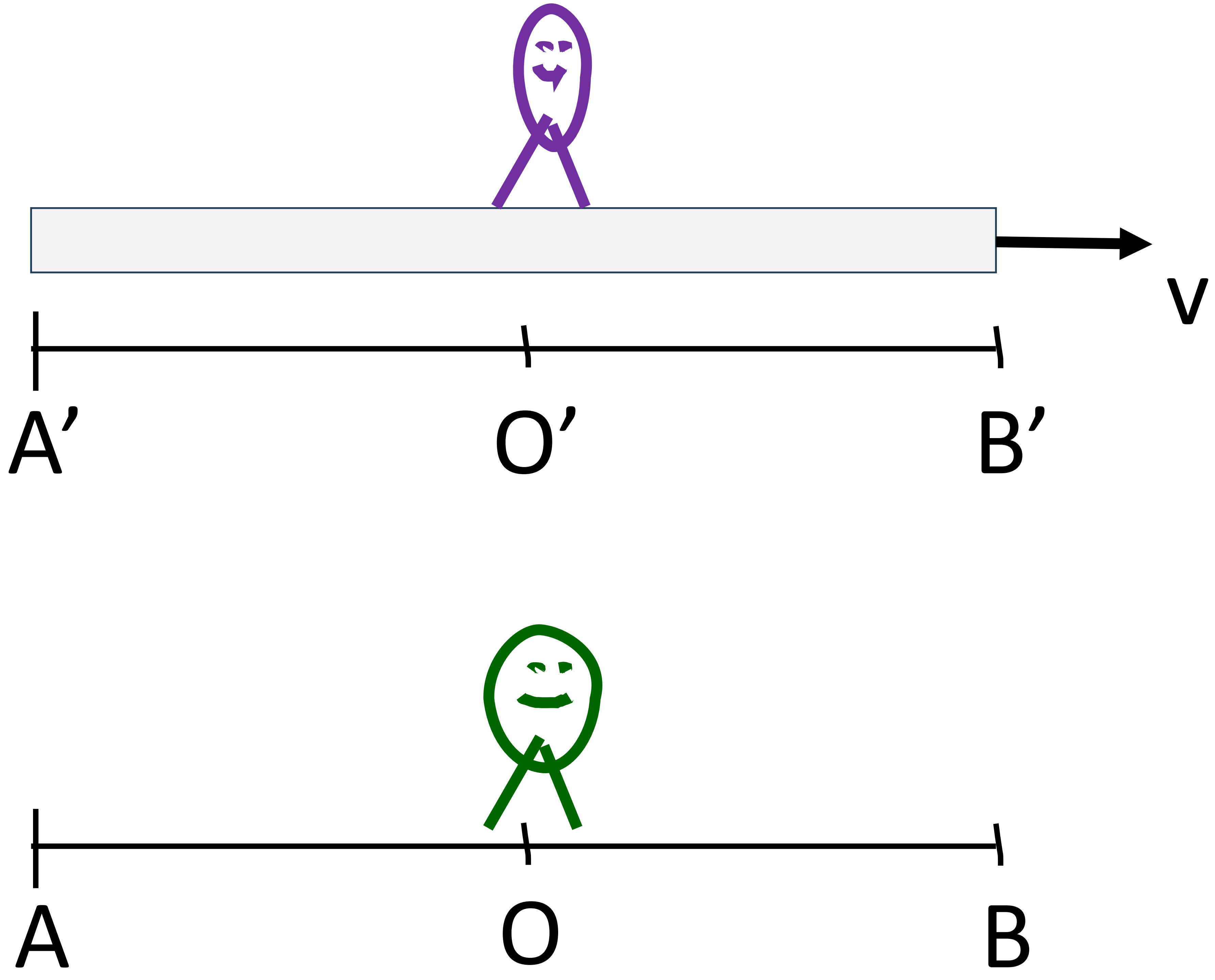
- “Gedanken” or thought-experiments to understand concepts
  - Light sparks/sources on moving vehicles, and
  - Observers inside or outside the vehicle (“not moving”)

SPECIAL RELATIVITY: RELATIVITY OF TIME

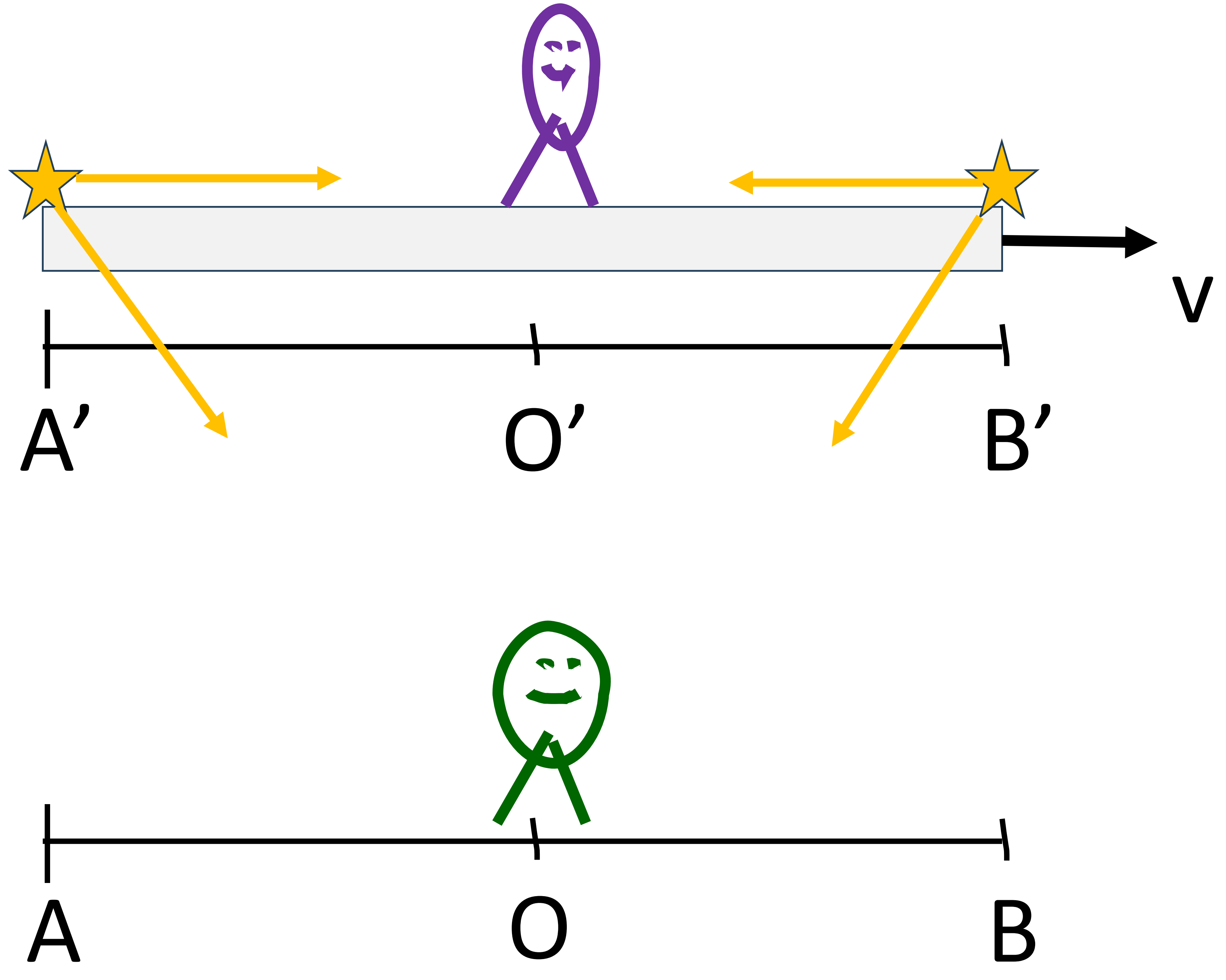
- Thought-experiment:



observer O’ in middle of moving platform

observer O in middle of platform but standing still

SPECIAL RELATIVITY: RELATIVITY OF TIME

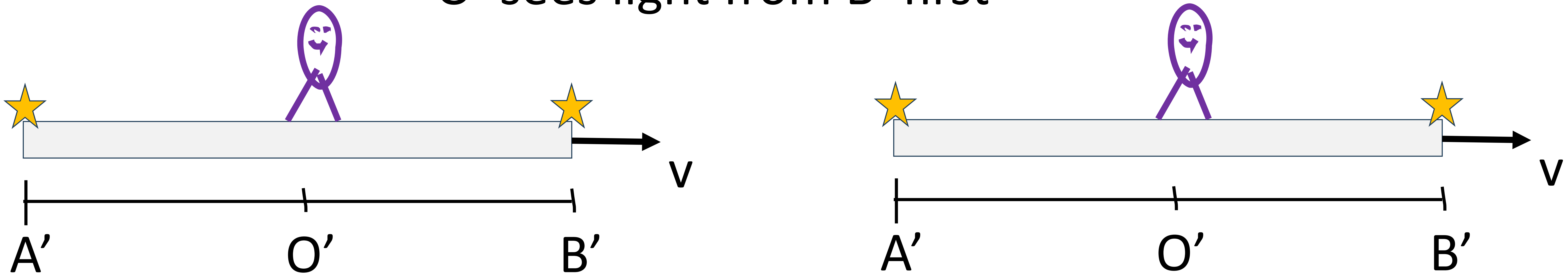


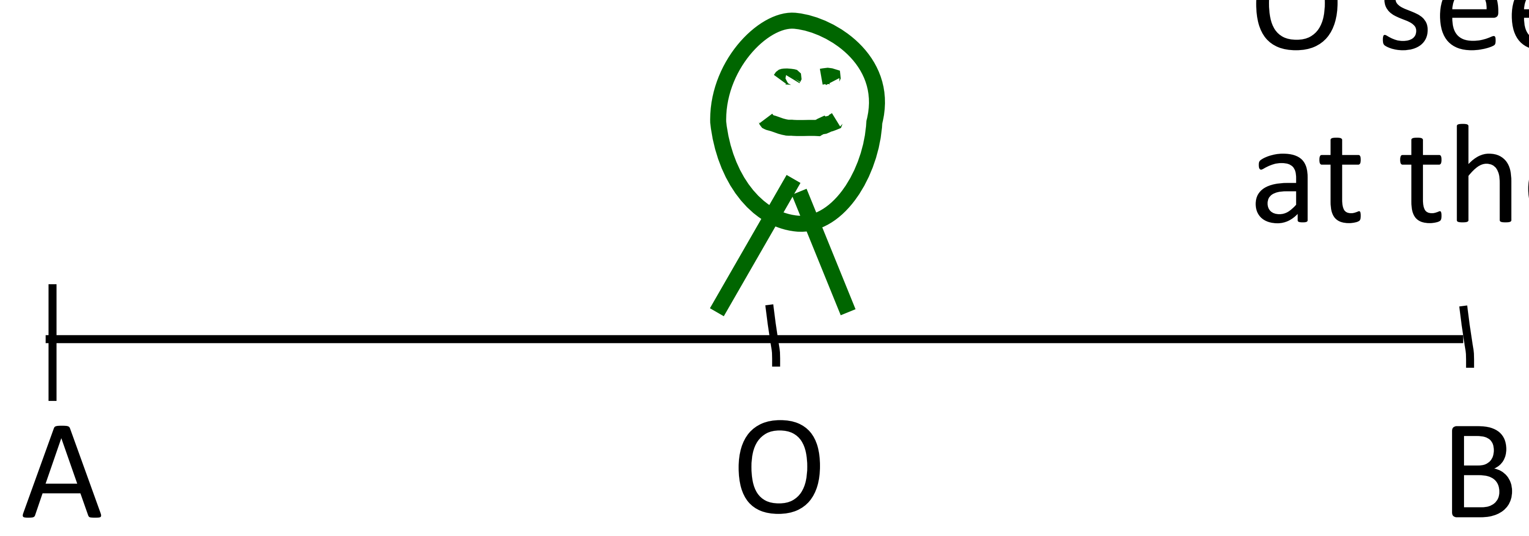
Light signals start from points A and B

To be seen by the observers in future

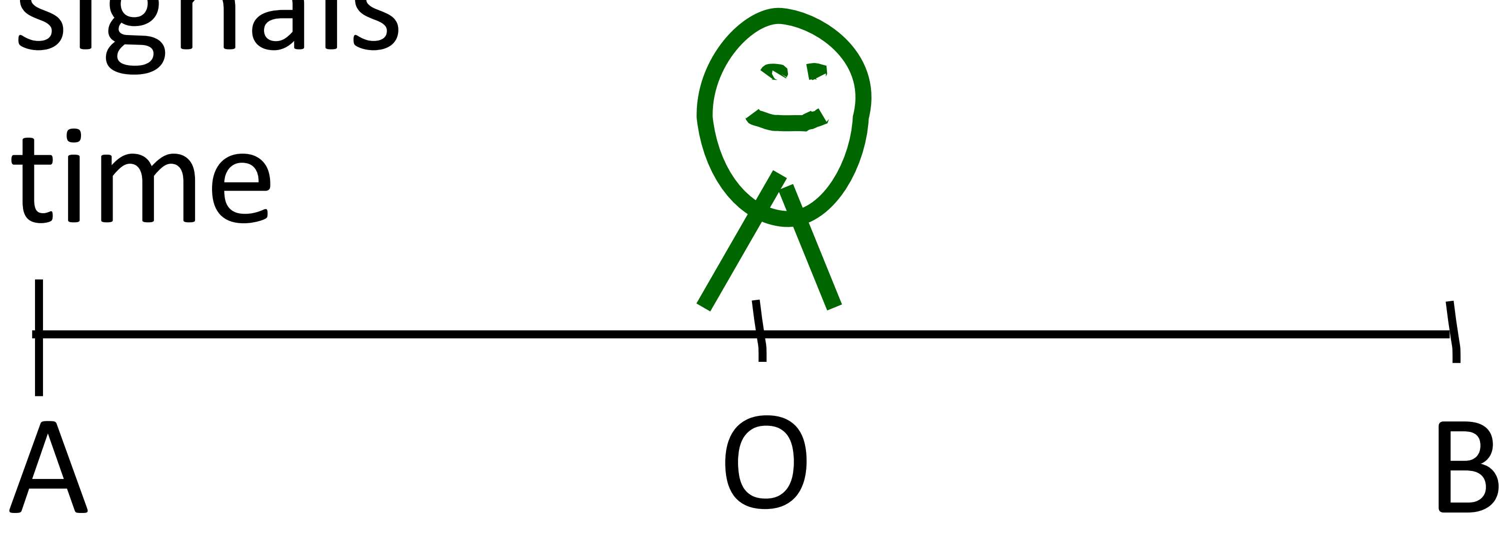
SPECIAL RELATIVITY: RELATIVITY OF TIME

O’ sees light from B’ first



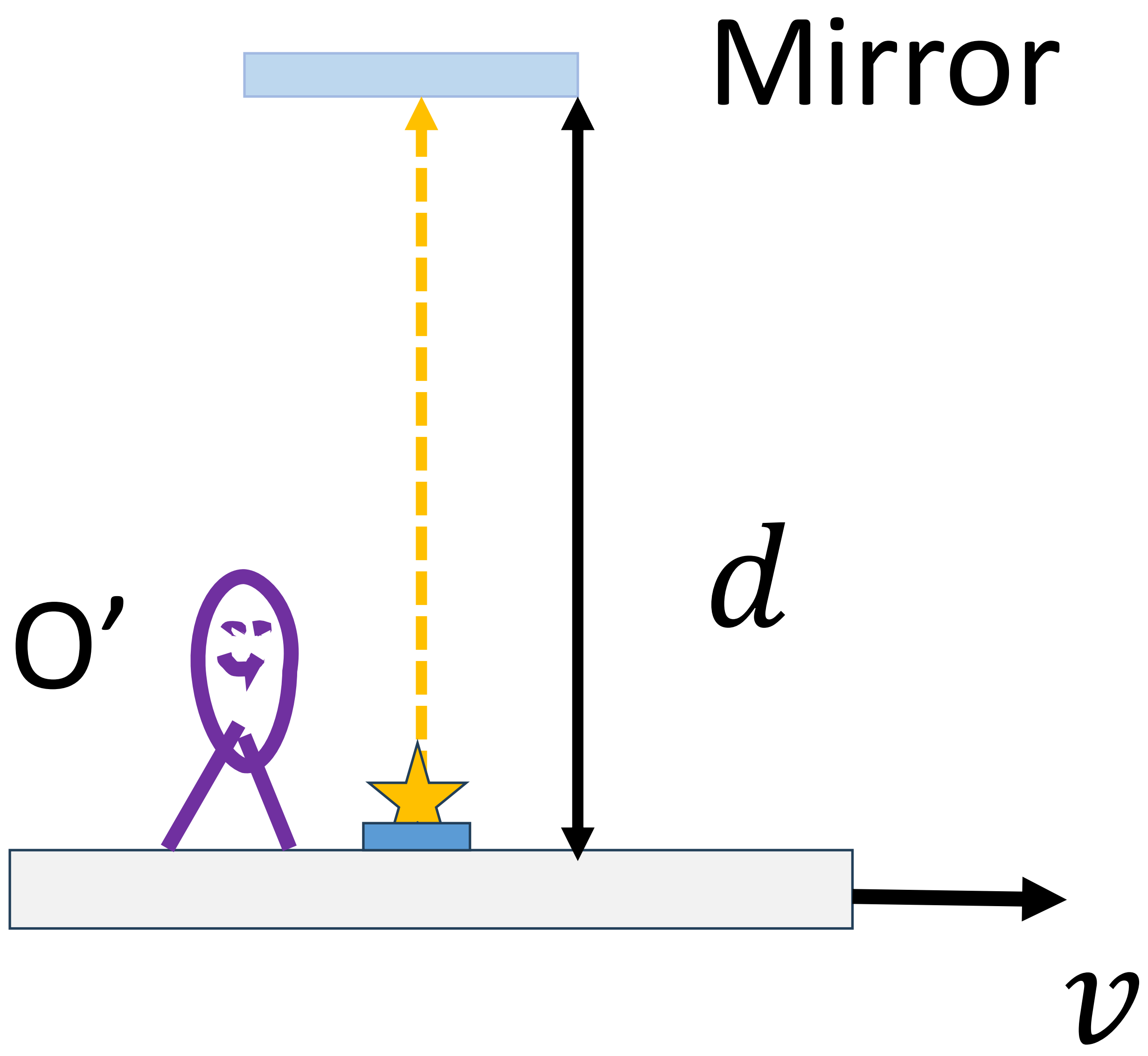


O sees both signals
at the same time



SPECIAL RELATIVITY: TIME DILATION

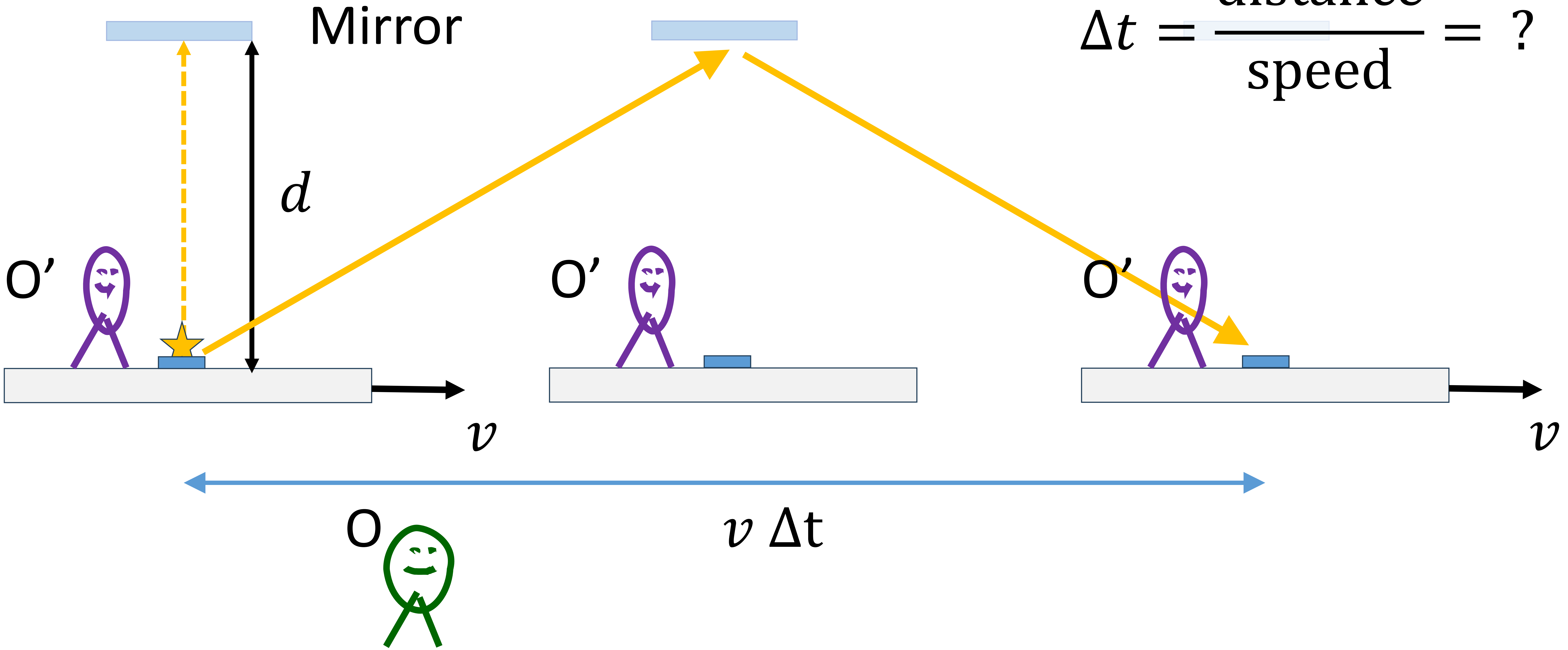
- Measuring time by light-pulse traveling forth and back



$$\Delta t' = \frac{\text{distance}}{\text{speed}} = \frac{2d}{c}$$

SPECIAL RELATIVITY: TIME DILATION

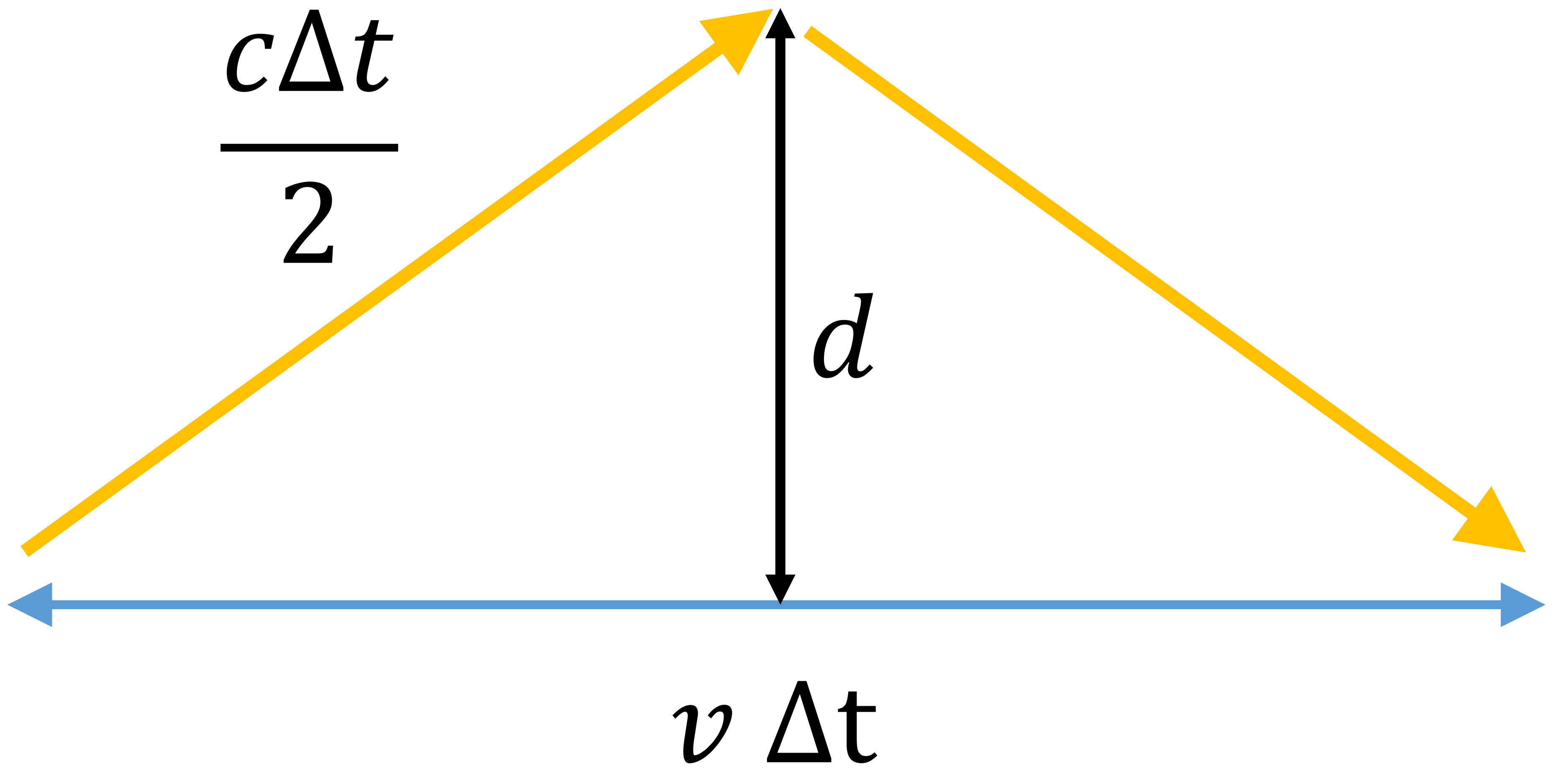
- Measuring time by traveling light-pulse



$$\Delta t = \frac{\text{distance}}{\text{speed}} = \ ?$$

SPECIAL RELATIVITY: TIME DILATION

- Measuring time by traveling light-pulse

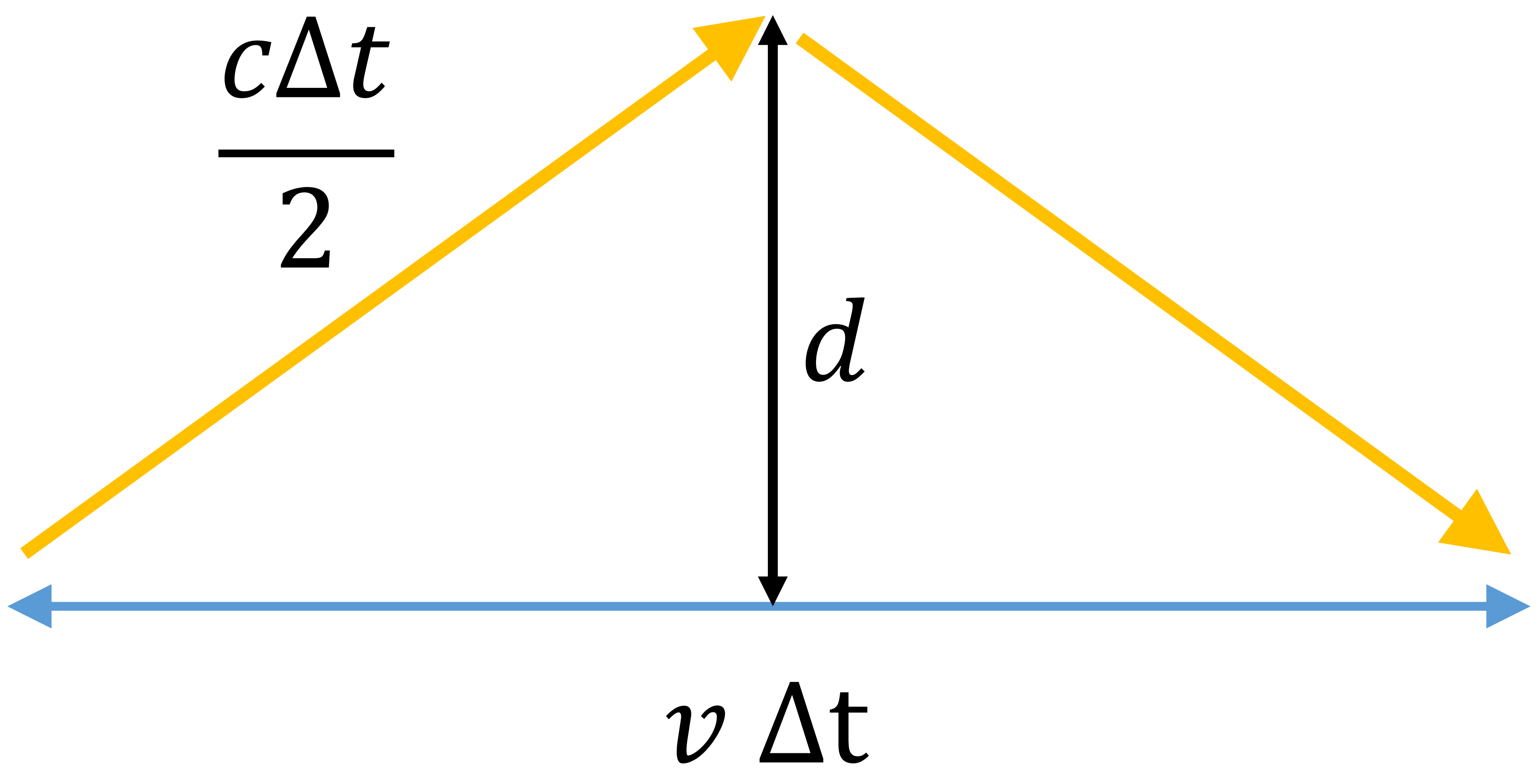


$$\Delta t = \frac{\text{distance}}{\text{speed}} = \;?$$

$$\left(\frac{v\Delta t}{2}\right)^2 + d^2 = \left(\frac{c\Delta t}{2}\right)^2$$

$$\Rightarrow \quad \Delta t = \frac{2\text{d}}{c\sqrt{1-\frac{\text{v}^2}{c^2}}} = \gamma \Delta t'$$

- Measuring time by traveling light-pulse



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SPECIAL RELATIVITY: TIME DILATION

- Time dilation

$$\Delta t = \frac{2\mathrm{d}}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = \gamma \Delta t'$$

- **Proper time frame**: events are happening at the **same position**

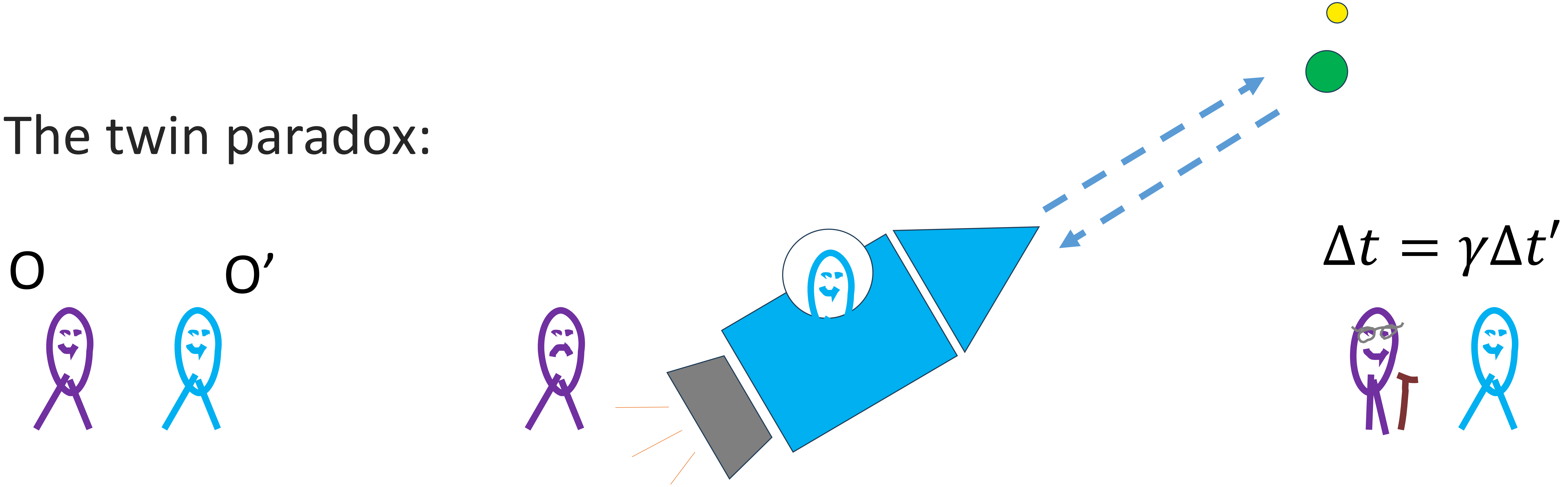
- **Time dilation slows down all processes:** mechanical, chemical, biological processes, etc.

SPECIAL RELATIVITY: THE TWIN PARADOX

- Time dilation

$$\Delta t = \frac{2\text{d}}{c\sqrt{1 - \frac{\text{v}^2}{c^2}}} = \gamma \Delta t'$$

- The twin paradox:



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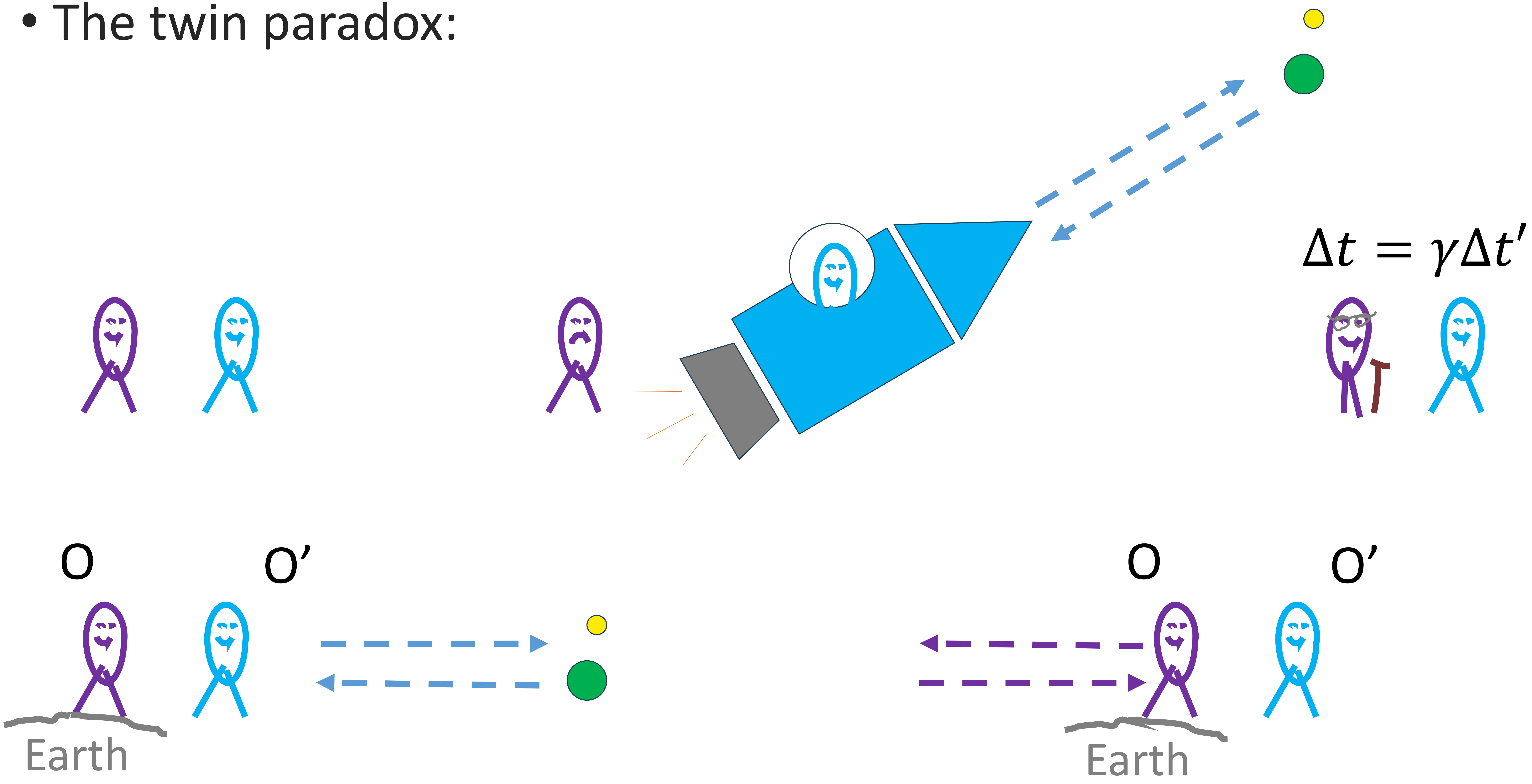
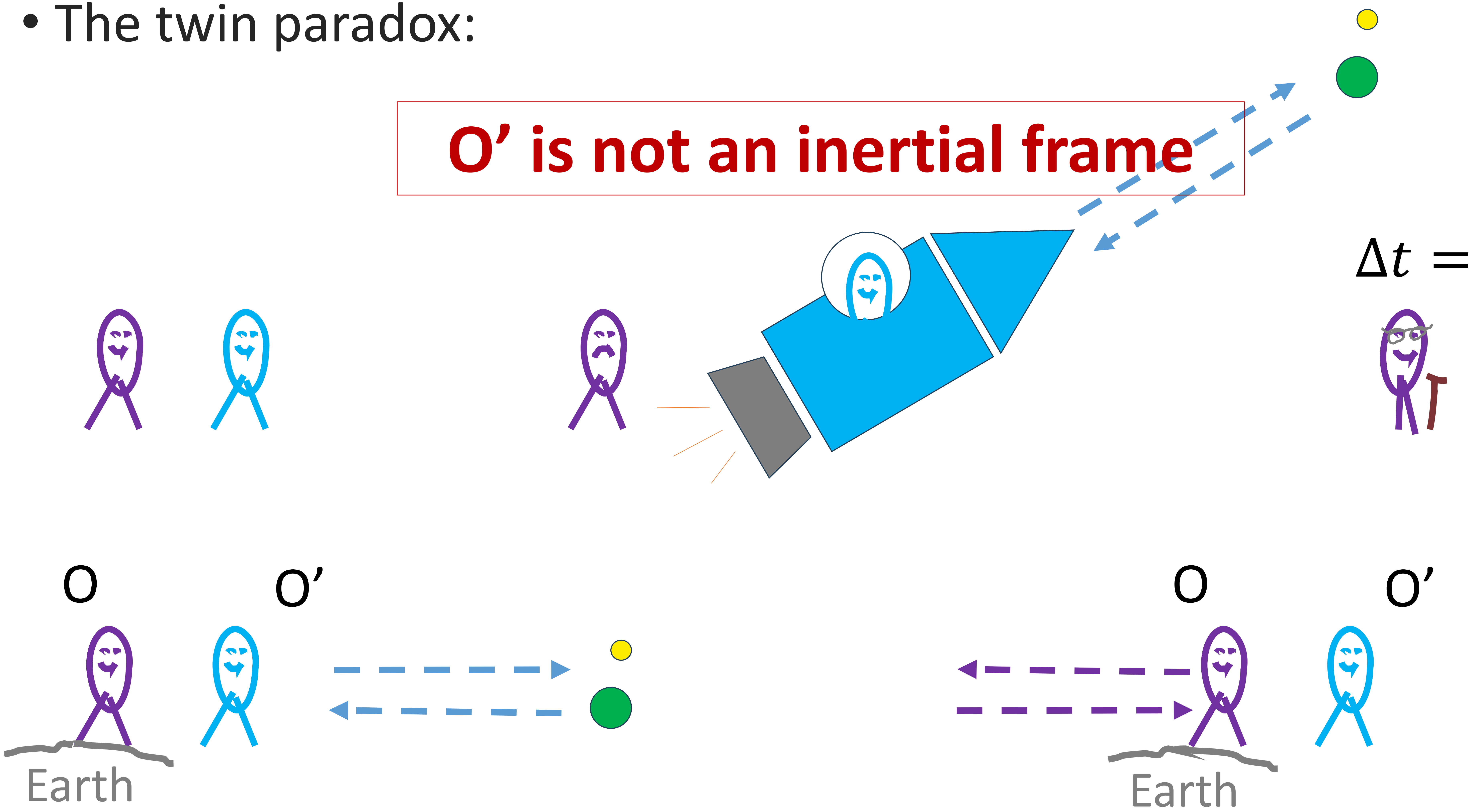


SPECIAL RELATIVITY: THE TWIN PARADOX

- The twin paradox:

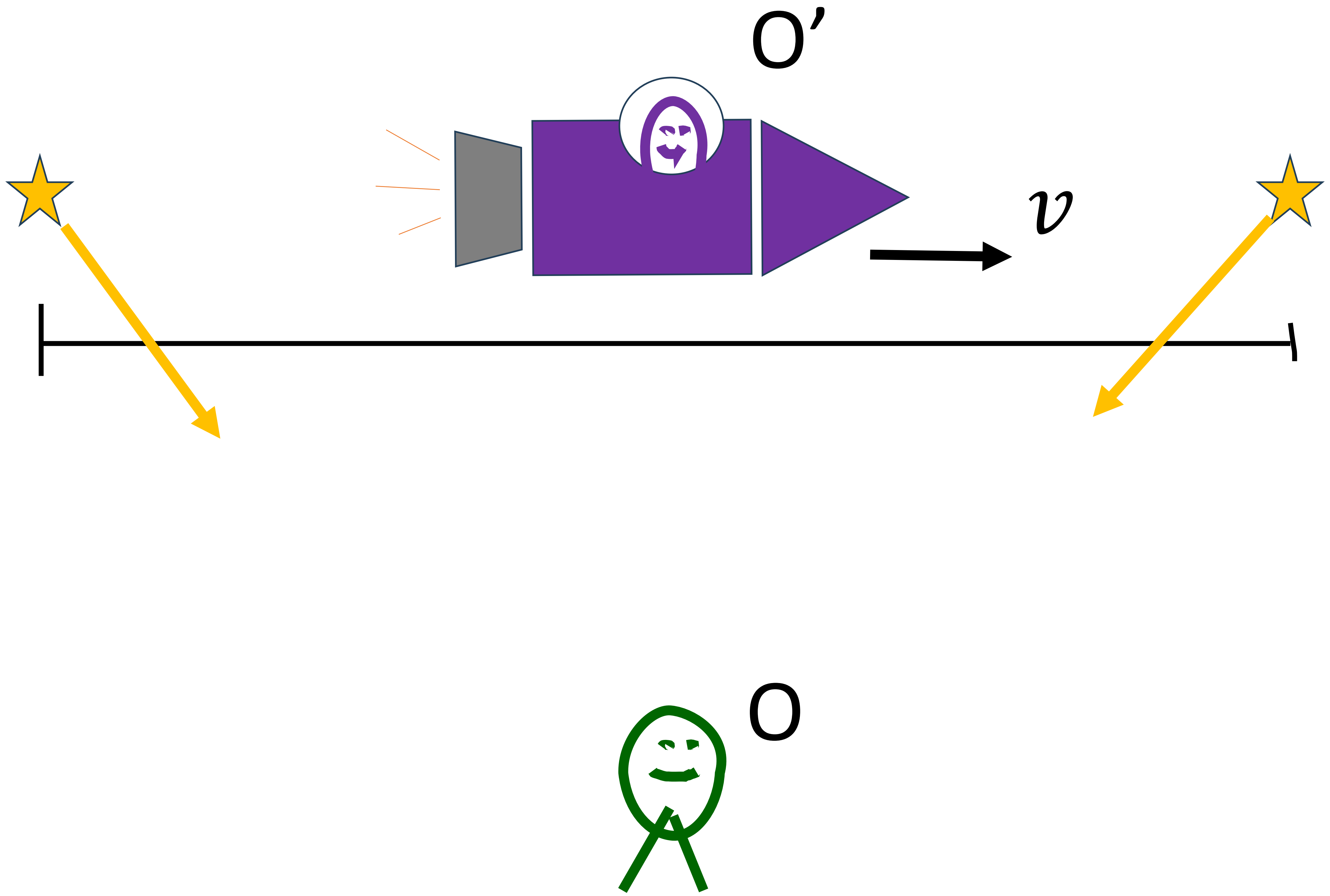
O’ is not an inertial frame



$$\Delta t = \gamma \Delta t'$$

SPECIAL RELATIVITY: LENGTH CONTRACTION

- **Proper length**: length object measured in frame where the object is in rest



Distance: $L = v\,\Delta t_p = \frac{v}{\gamma}\Delta t$

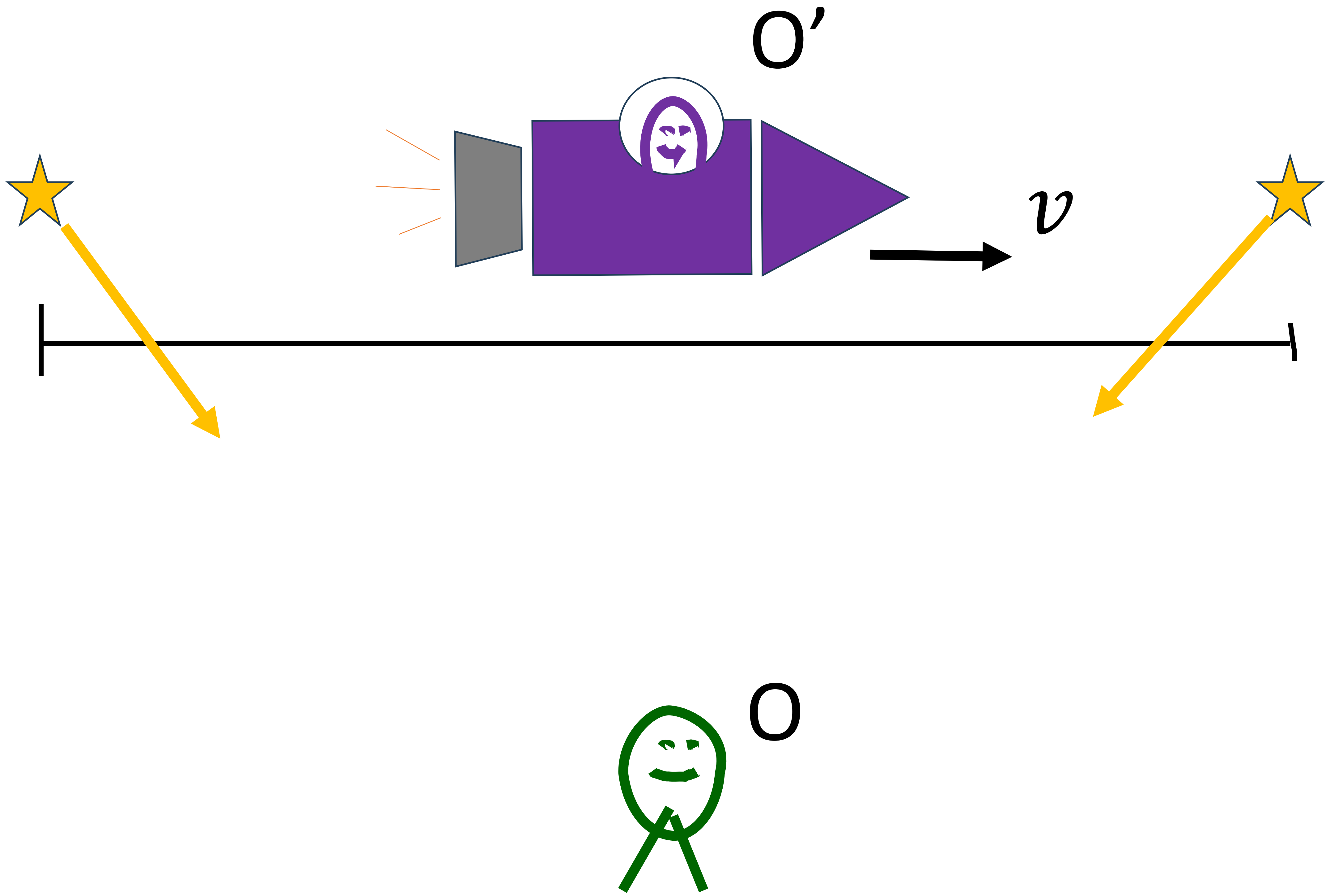
Proper time: $\Delta t_p = \frac{\Delta t}{\gamma}$

Proper length: L_p

Time: $\Delta t = L_p/v$

SPECIAL RELATIVITY: LENGTH CONTRACTION

- **Proper length**: length object measured in frame where the object is in rest



Distance: $L = v\,\Delta t_p = \frac{v}{\gamma}\Delta t$

Proper time: $\Delta t_p = \frac{\Delta t}{\gamma}$

$$\Longrightarrow \quad \boxed{L = \frac{L_p}{\gamma} = L_p\sqrt{1 - \frac{v^2}{c^2}}}$$

Proper length: L_p

Time: $\Delta t = L_p / v$

SPECIAL RELATIVITY: RELATIVISTIC DOPPLER EFFECT

Classical Doppler effect

$$f' = \left(\frac{v + v_O}{v - v_S}\right) f$$

- Velocity observer: v_O
- Velocity source: v_S
- Wavelength: $\lambda = \frac{v}{f}$

SPECIAL RELATIVITY: RELATIVISTIC DOPPLER EFFECT

Classical Doppler effect

$$f' = \left(\frac{v + v_O}{v - v_S}\right) f$$

- Velocity observer: v_O
- Velocity source: v_S
- Wavelength: $\lambda = \frac{v}{f}$

Relativistic Doppler effect

$$f' = \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}}\, f$$

- Observer and source approach each other with velocity: v
- Only depends on relative velocity
- Wavelength: $\lambda = \frac{v}{f}$

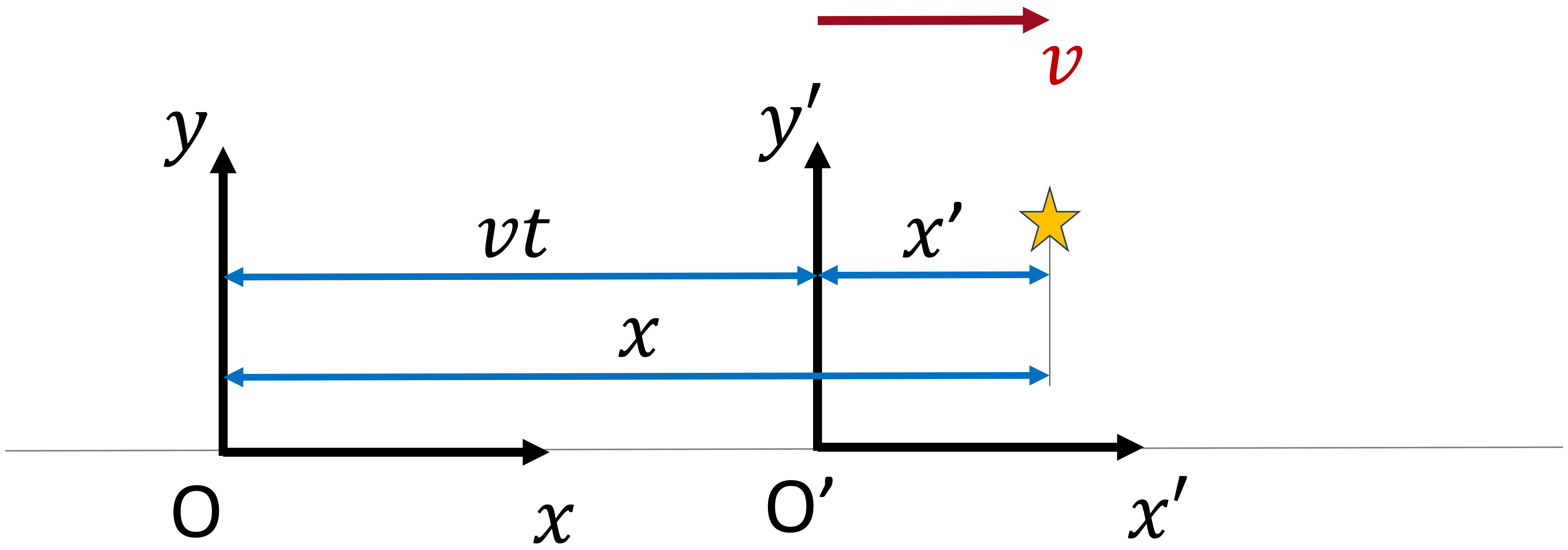
SPECIAL RELATIVITY: THE LORENTZ TRANSFORMATION

(Classical) Galilean space-time transformation:

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

(Relativistic) Lorentz space-time transformation:

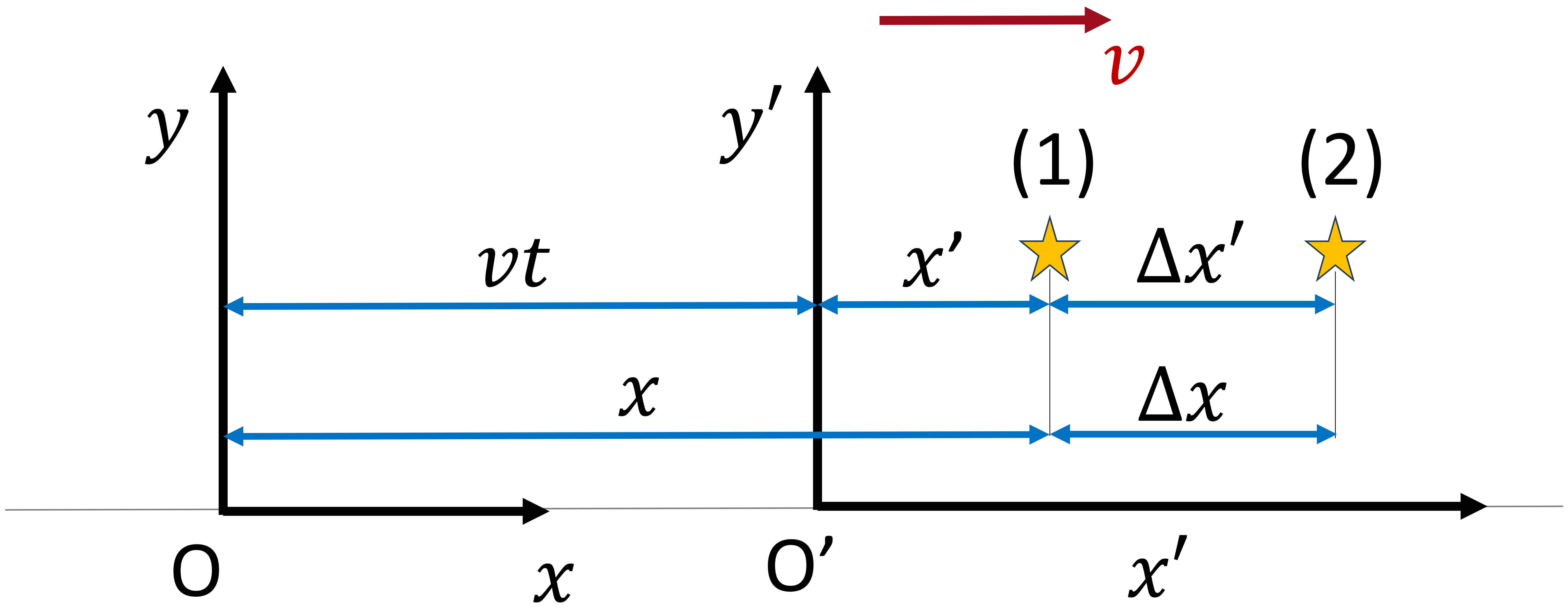
? ? ?



SPECIAL RELATIVITY: THE LORENTZ TRANSFORMATION

(Relativistic) **Lorentz space-time transformation:**

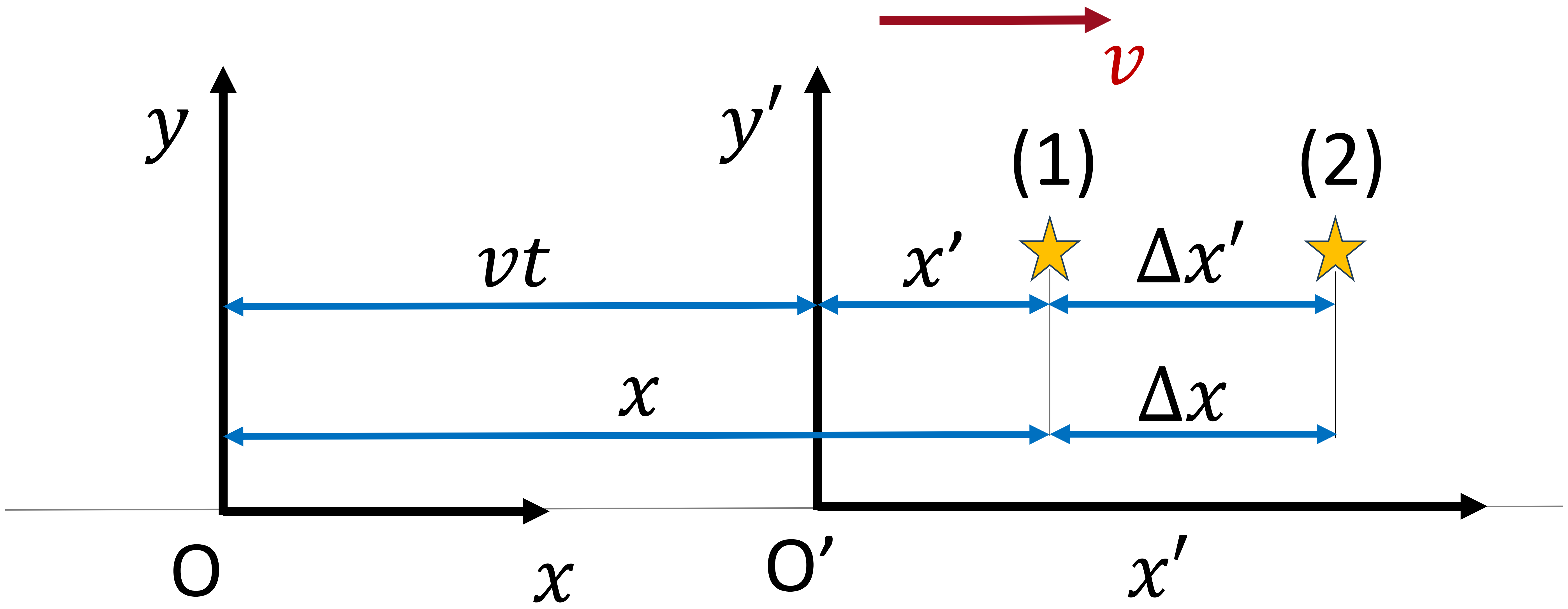
$$x' = \gamma\,(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$



SPECIAL RELATIVITY: THE LORENTZ TRANSFORMATION

(Relativistic) Lorentz space-time transformation: **simultaneity**

$$\Delta x' = \gamma\,(\Delta x - v\Delta t),\; y' = y,\; z' = z, \quad \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$



SPECIAL RELATIVITY: LORENTZ VELOCITY TRANSFORMATION

$$\Delta x' = \gamma\,(\Delta x - v\Delta t),\; y' = y,\; z' = z, \quad \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

- Velocity $u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}\,dx\right)} = \frac{\frac{dx}{dt} - v}{1 - \frac{v\,dx}{c^2\,dt}} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$

SPECIAL RELATIVITY: LORENTZ VELOCITY TRANSFORMATION

$$\Delta x' = \gamma\,(\Delta x - v\Delta t),\quad y' = y,\ z' = z,\quad \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

- Velocity $u'_x = \dfrac{dx'}{dt'} = \dfrac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \dfrac{\frac{dx}{dt} - v}{1 - \frac{v\,dx}{c^2\,dt}} = \dfrac{u_x - v}{1 - \frac{vu_x}{c^2}}$

- Velocity $u'_y = \dfrac{dy'}{dt'} = \dfrac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \dfrac{\frac{dy}{dt}}{\gamma\left(1 - \frac{v\,dx}{c^2\,dt}\right)} = \dfrac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$

SPECIAL RELATIVITY: LORENTZ VELOCITY TRANSFORMATION

(Relativistic) Lorentz velocity transformation

$$\boldsymbol{u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}}, \quad u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}, \quad u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

- Limit $v \to 0 \quad \Longrightarrow \quad u'_x = u_x - v$ Galilean transformation
- Limit $u_x \to c \quad \Longrightarrow \quad u'_x = \frac{c-v}{1-\frac{cv}{c^2}} = c$ Constant velocity c

SPECIAL RELATIVITY: THE LORENTZ TRANSFORMATION

Lorentz **space-time** transformation:

$$x' = \gamma\,(x - vt), \quad y' = y, \quad z' = z, \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Lorentz **velocity** transformation:

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}, \quad u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}, \quad u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

SPECIAL RELATIVITY: INVARIANTS

- **Invariant** quantities are the **same for all observers**
- Invariant under Galilean transformations:
  - Time, time-intervals, simultaneity of events
  - Length and distance
  - BUT NOT: Energy, momentum, velocity, position, …

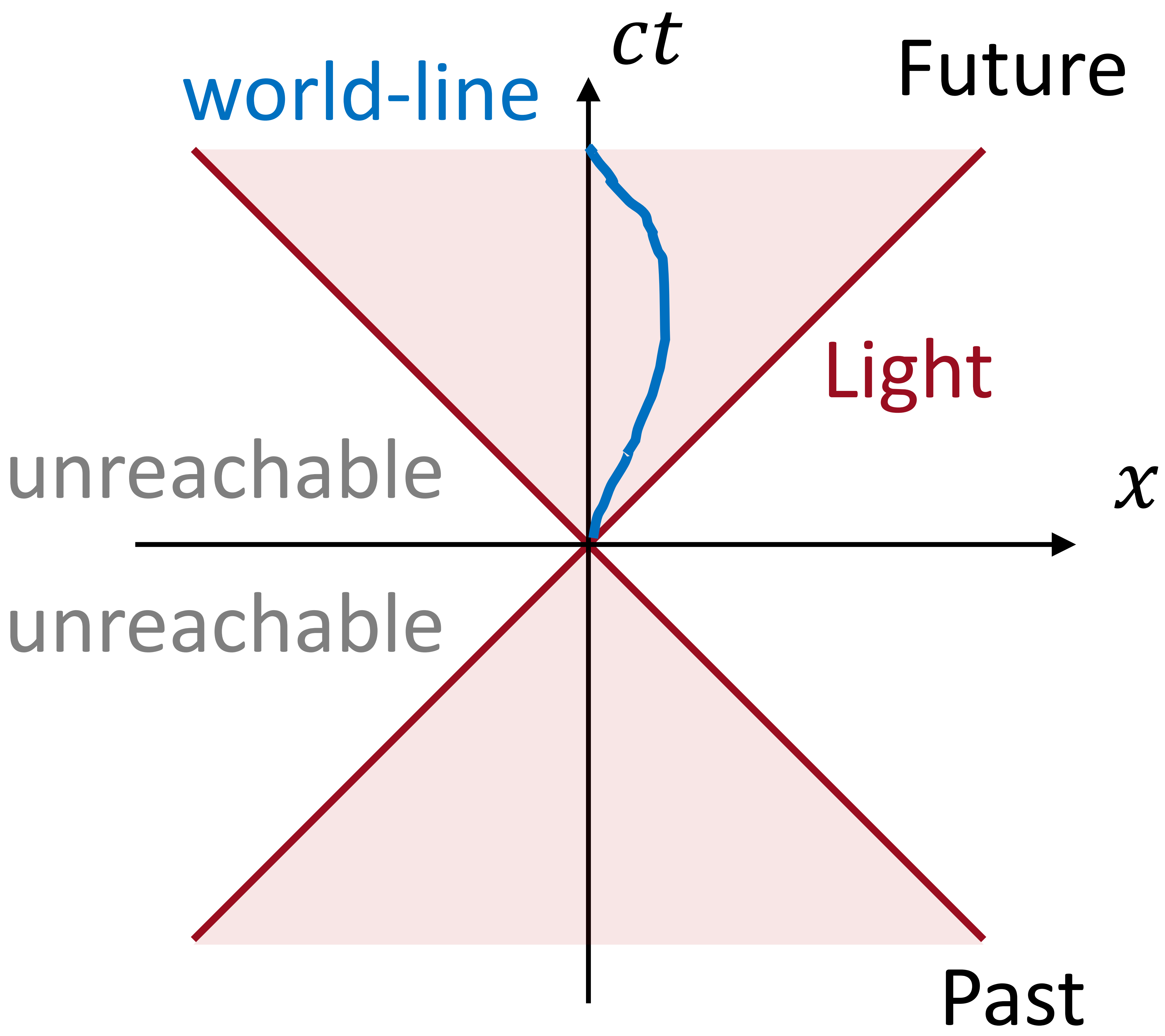
- What is **invariant** under the Lorentz transformation?
  - The distance in Minkowski space-time is **Lorentz invariant**:

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$-c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

SPECIAL RELATIVITY: SPACE-TIME GRAPHS

- **Space-time graph:** x vs. t
- **World-line:** Path in space-time
- Derivative of path $\frac{1}{c}\frac{dx}{dt} \leq 1$



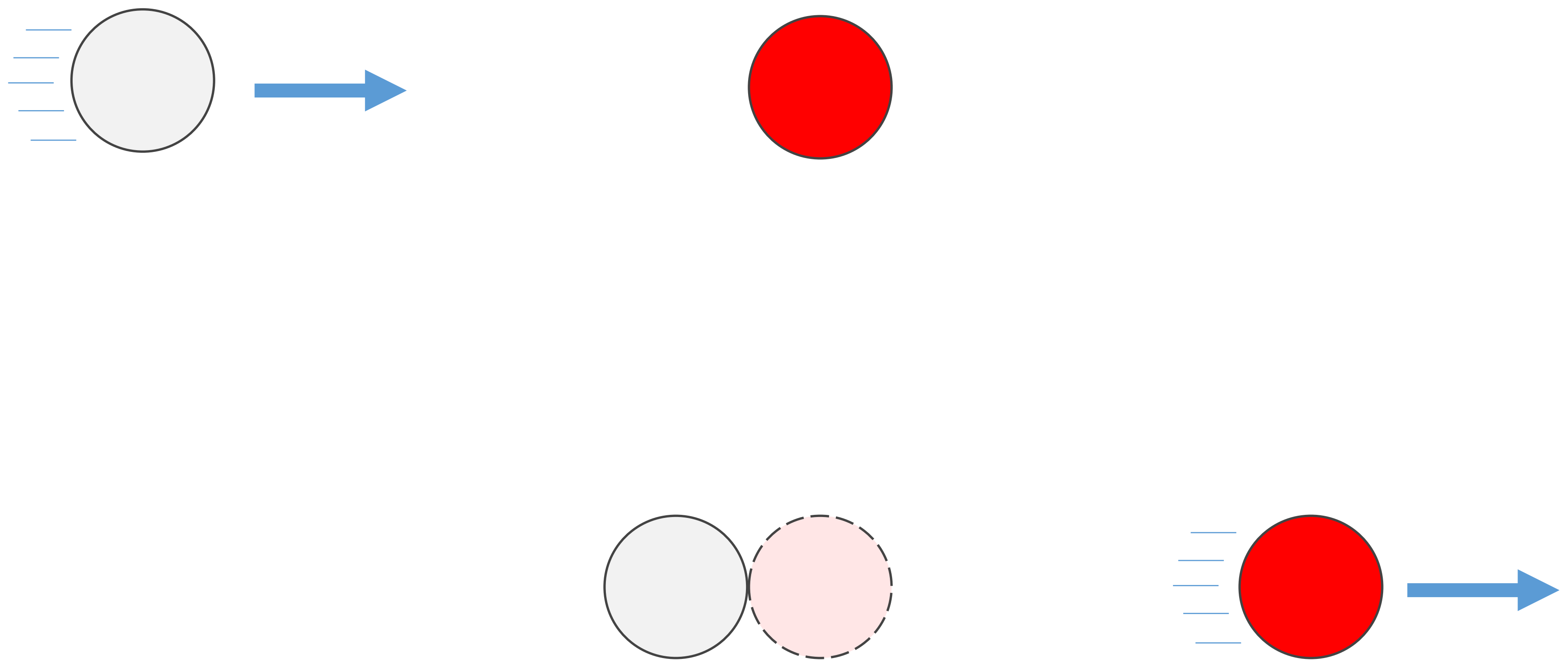
SUMMARY LORENTZ TRANSFORMATIONS

- Galilean relativity is not true
  - Time is **not** a universal parameter
  - Constant maximum velocity c
  - Length contraction and time dilation
- **Proper time**: frame where events at the same position
- **Proper length**: frame where object is in rest
- Lorentz transformations instead of Galilean
  - Lorentz **space-time** transformations
  - Lorentz **velocity** transformations

CLASSICAL BILLIARD: GALILEAN LINEAR MOMENTUM

$$\boldsymbol{p_1 + p_2 = \tilde{p}_1 + \tilde{p}_2}$$

$p_1 = mu_1$ $\quad$ $p_2 = 0$



$\tilde{p}_1 = 0$ $\quad$ $\tilde{p}_2 = m\tilde{u}_2$

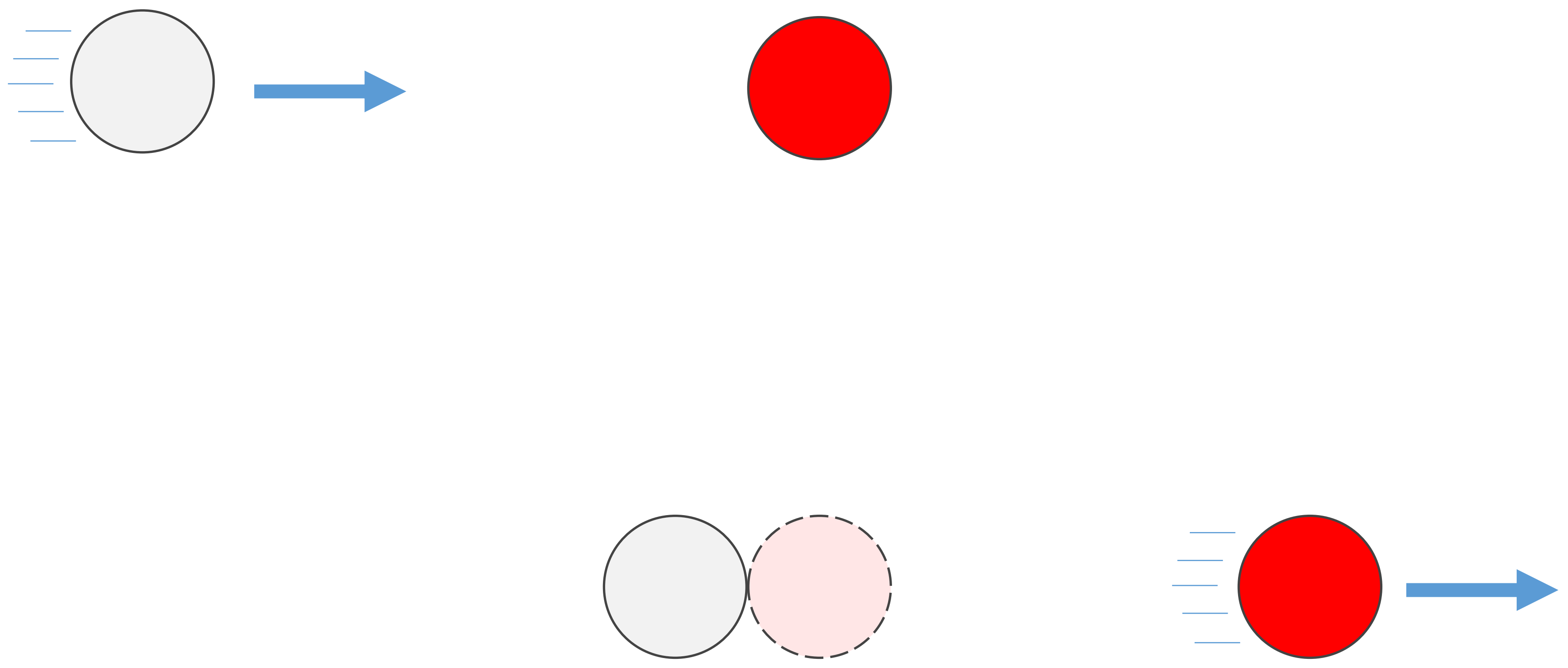
- Total linear momentum $\vec{P}$ should be conserved
- Before and after collision
- In the frame of a pool table red ball is standing still before being hit

CLASSICAL BILLIARD: GALILEAN LINEAR MOMENTUM

Pool table frame

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \widetilde{\boldsymbol{p}}_1 + \widetilde{\boldsymbol{p}}_2$$

$p_1 = mu_1$ $\quad$ $p_2 = 0$

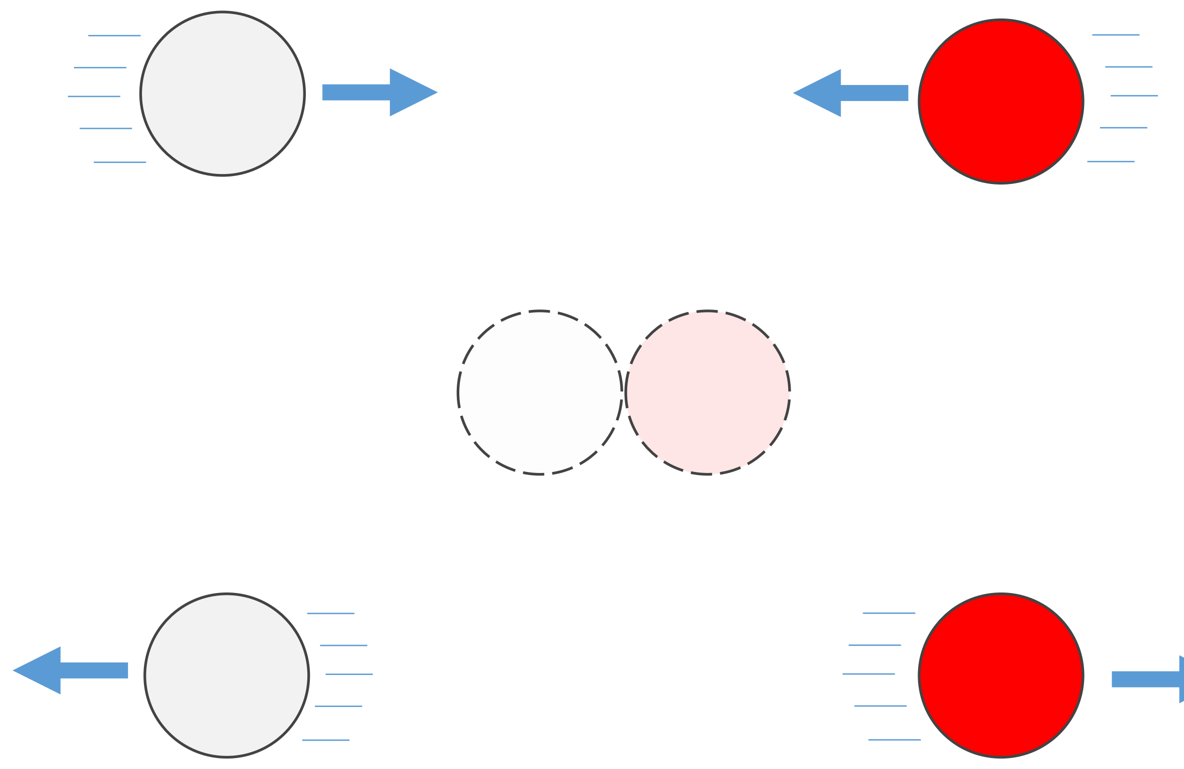


$\tilde{p}_1 = 0$ $\quad$ $\tilde{p}_2 = m\tilde{u}_2$

Center of mass frame

$$\boldsymbol{p'}_1 + \boldsymbol{p'}_2 = \widetilde{\boldsymbol{p}}'_1 + \widetilde{\boldsymbol{p}}'_2$$

$p'_1 = mu'_1$ $\quad$ $p'_2 = mu'_2$



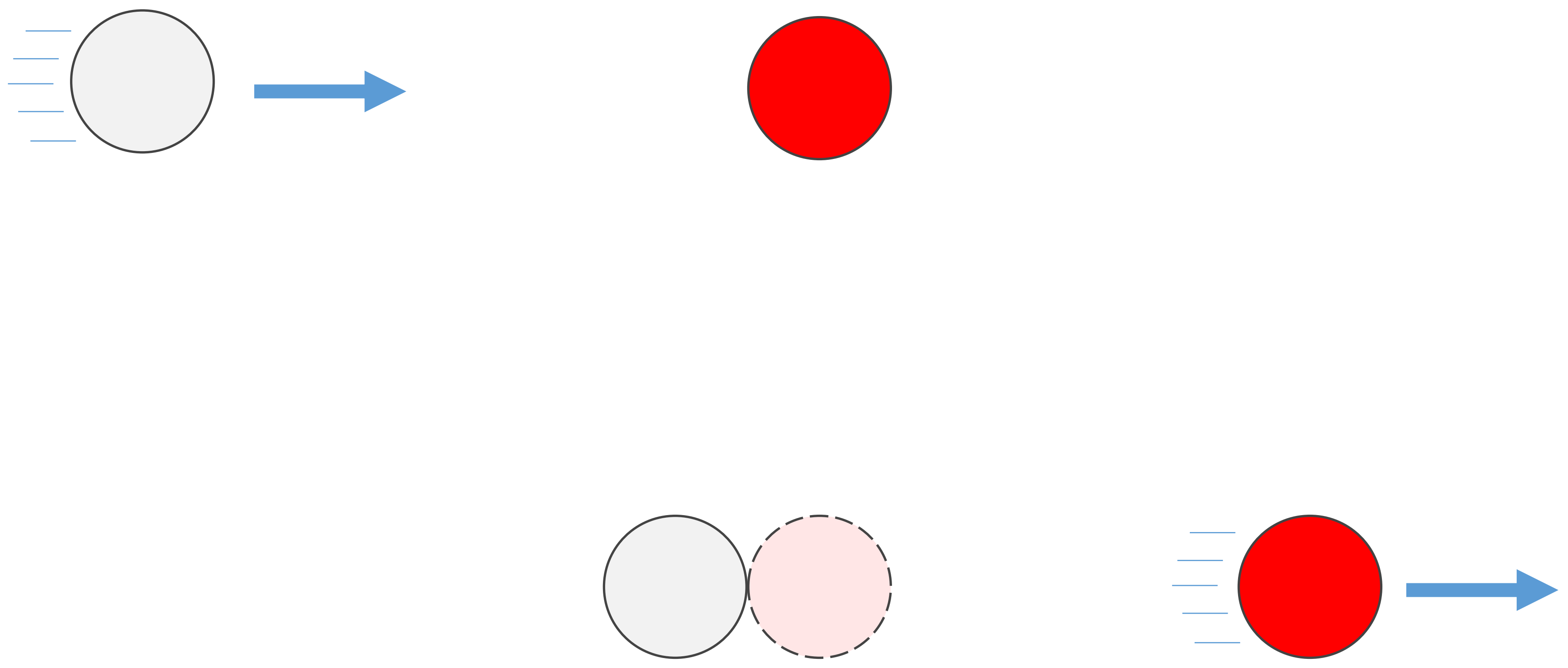
$\tilde{p}'_1 = m\tilde{u}'_1$ $\quad$ $\tilde{p}'_2 = m\tilde{u}'_2$

CLASSICAL BILLIARD: GALILEAN LINEAR MOMENTUM

Pool table frame

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \tilde{\boldsymbol{p}}_1 + \tilde{\boldsymbol{p}}_2$$

$p_1 = mu$ $\quad$ $p_2 = 0$

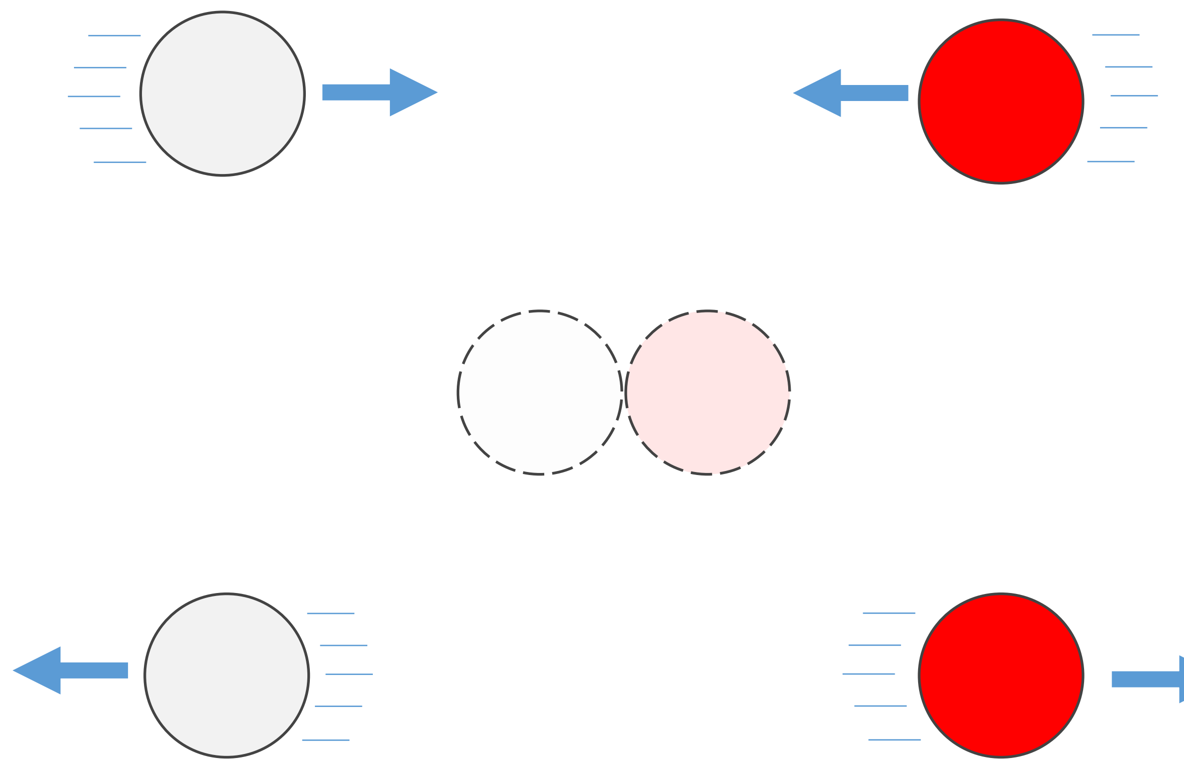


$\tilde{p}_1 = 0$ $\quad$ $\tilde{p}_2 = mu$

Center of mass frame

$$\boldsymbol{p'}_1 + \boldsymbol{p'}_2 = \tilde{\boldsymbol{p}}'_1 + \tilde{\boldsymbol{p}}'_2$$

$p'_1 = mu/2$ $\quad$ $p'_2 = -mu/2$



$\tilde{p}'_1 = -mu/2$ $\quad$ $\tilde{p}'_2 = mu/2$

SPECIAL RELATIVITY: RELATIVISTIC LINEAR MOMENTUM

Lorentz velocity transformation:

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}, \quad u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}, \quad u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

- Linear momentum $\vec{p}$ should be conserved: $\vec{p} \neq m\vec{u}$
- New definition for linear momentum:

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\, m\vec{u}$$

SPECIAL RELATIVITY: RELATIVISTIC LINEAR MOMENTUM

- New definition for linear momentum:

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\, m\vec{u}$$

- Momentum grows faster than linear with velocity
- Adding extra momentum by a force is harder for faster particles:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(\gamma m\vec{u})}{dt}$$

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- Adding extra momentum by a force is harder for faster particles:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d\vec{u}}{dt} = \gamma\, m \frac{d\vec{u}}{dt}$$

- We need also a new definition of kinetic energy of a particle

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- Adding extra momentum by a force is harder for faster particles, in 1D:

$$F = \frac{dp}{dt} = \frac{d\left\{mu\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\right\}}{dt} = m\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}}\frac{du}{dt}$$

- Kinetic energy of a particle as "work" W done by force $\vec{F}$

$$W = \int_A^B F\,dx = \int_A^B \frac{dp}{dt}dx = \int_A^B m\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}}\frac{du}{dt}dx$$

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- Kinetic energy of a particle as "work" W done by force $\vec{F}$

$$W = \int_A^B F\,dx = \int_A^B \frac{dp}{dt}\,dx = \int_A^B m\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}} \frac{du}{dt}\,dx$$

$$= \int_0^t m\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}} \frac{du}{dt}\,(u\,dt)$$

$$= \int_0^u m\,u\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}} du$$

$dx = u\,dt$

$u = 0 \rightarrow u$

- Solve the integral by substitution

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- Solve by substitution: $z \leftarrow 1 - \frac{u^2}{c^2}$ and $dz \leftarrow -\frac{2u}{c^2} du$

$$\int_0^u m\, u \left(1 - \frac{u^2}{c^2}\right)^{-\frac{3}{2}} du = \int_1^{1-\frac{u^2}{c^2}} m\, z^{-\frac{3}{2}} \frac{c^2}{2}\, dz$$

$$= mc^2 \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} - mc^2$$

$$\Rightarrow \quad K = (\gamma - 1)\, mc^2$$

- Kinetic energy increases nonlinear with velocity

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- For small velocities $u \ll c$ we recover the classical energy:

$$K = mc^2\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} - mc^2 \approx mc^2\left(1 + \frac{u^2}{2c^2}\right) - mc^2 = \frac{mu^2}{2}$$

- For velocities $u \to c$ the relativistic energy goes to infinity:

$$K = \lim_{u \to c} mc^2\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} - mc^2 \quad \to \quad +\infty$$

- For relativistic velocities in between, e.g. $u = 0.5\,c$ the relativistic energy goes up faster than classical (quadratic)

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- The kinetic energy has a term only depending on the mass:

$$K = \gamma mc^2 - mc^2$$

- We call this term the **rest energy**: $\mathrm{E_R} = mc^2$

- Total energy = kinetic energy + rest energy:

$$E = K + mc^2$$

- The total energy is: $E = \gamma mc^2 = \dfrac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}$

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- Relation between the total energy and the momentum

The total energy is: $E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}$

Momentum is: $p = \gamma mu$

$$\Longrightarrow \quad E^2 = p^2c^2 + (mc^2)^2$$

- For photons (massless) we obtain: $E = pc$

- Rest energies can be very large & are independent of frame

SPECIAL RELATIVITY: RELATIVISTIC ENERGY

- Rest energies can be very large & are independent of frame

Examples:

- Rest energy of an electron: $m_e c^2 = 0.511$ MeV
- Rest energy of a proton: $m_p c^2 = 938$ MeV

SUMMARY RELATIVISTIC MOMENTUM AND ENERGY

- Momentum p increases nonlinear with velocity
- Kinetic energy also increases nonlinear with velocity
- Rest energy $E_R = mc^2$
- Total energy = kinetic energy + rest energy

GENERAL RELATIVITY

- General relativity merges the concepts of inertia and gravitation

  - Inertia is the resistance to be accelerated by a force

$$\vec{F} = m_i \, \vec{a}$$

  - Gravitation is also proportional to the same mass: $m_g = m_i$

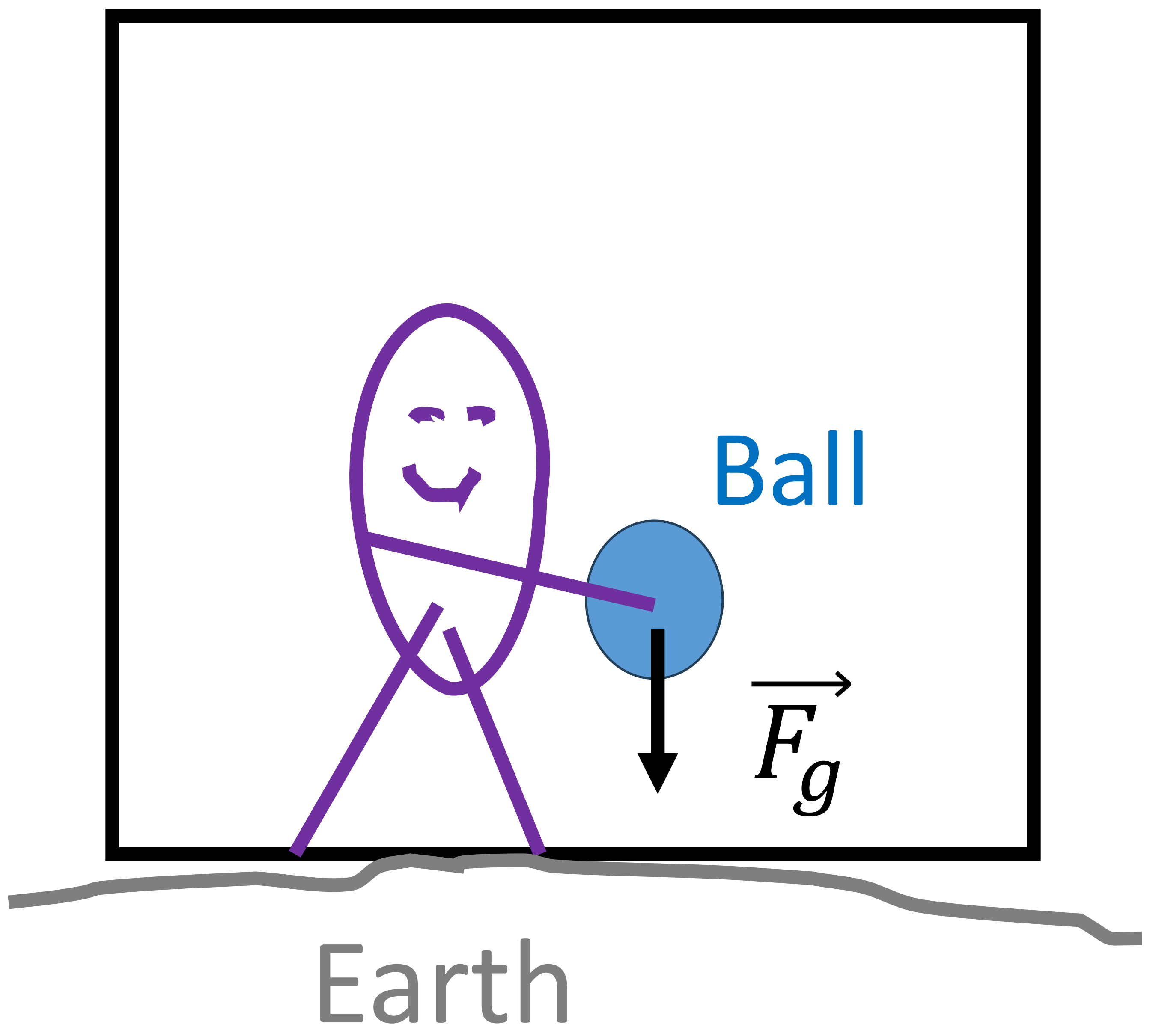
$$\overrightarrow{F_g} = G \, \frac{m_g m_{\mathrm{Earth}}}{r^2} \propto m_g$$

  - In general relativity both are connected: mass curves space

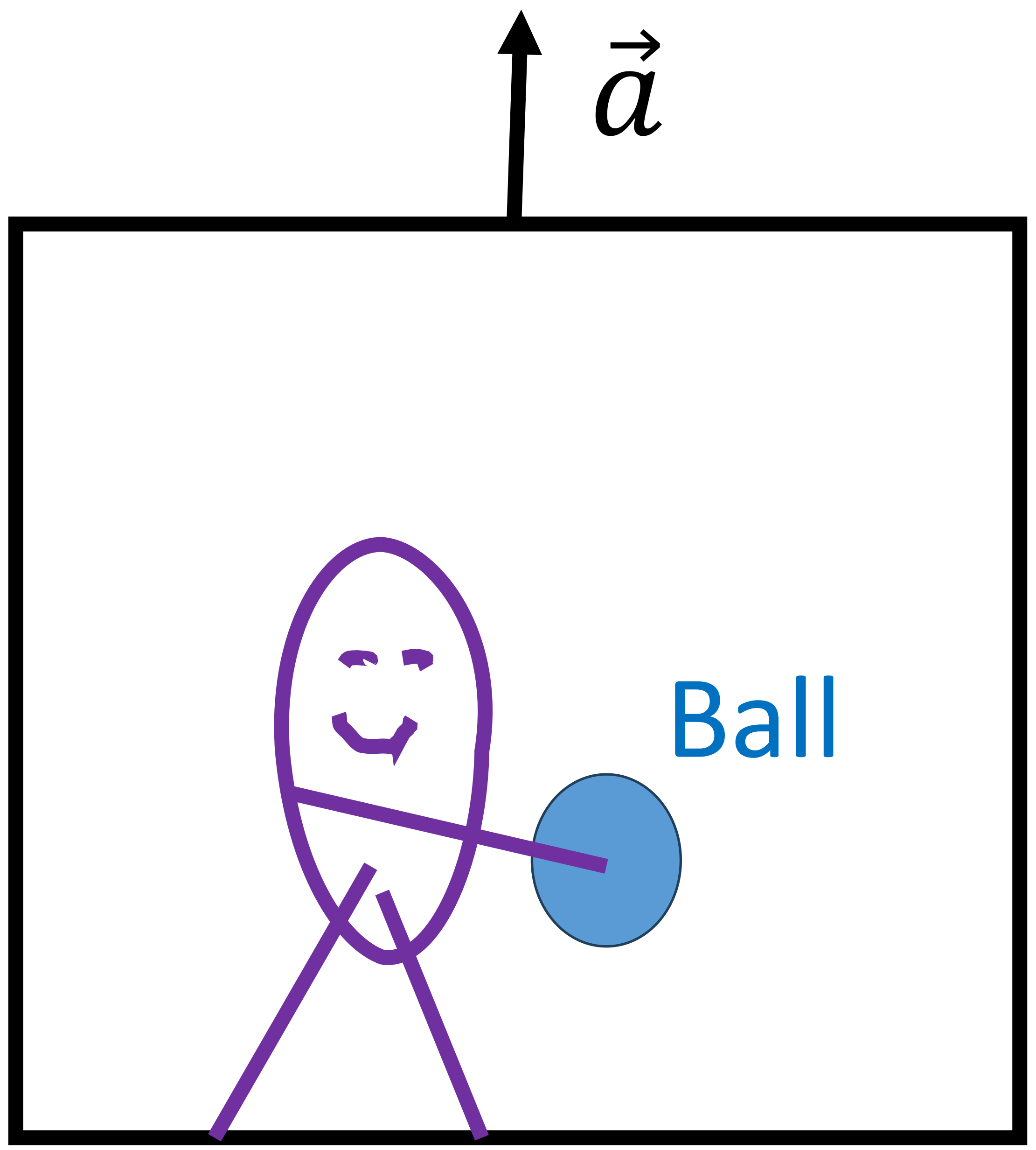
GENERAL RELATIVITY

- Thought experiment: For the observer in a closed box the two situations are equal:

(1) Ball falls towards Earth



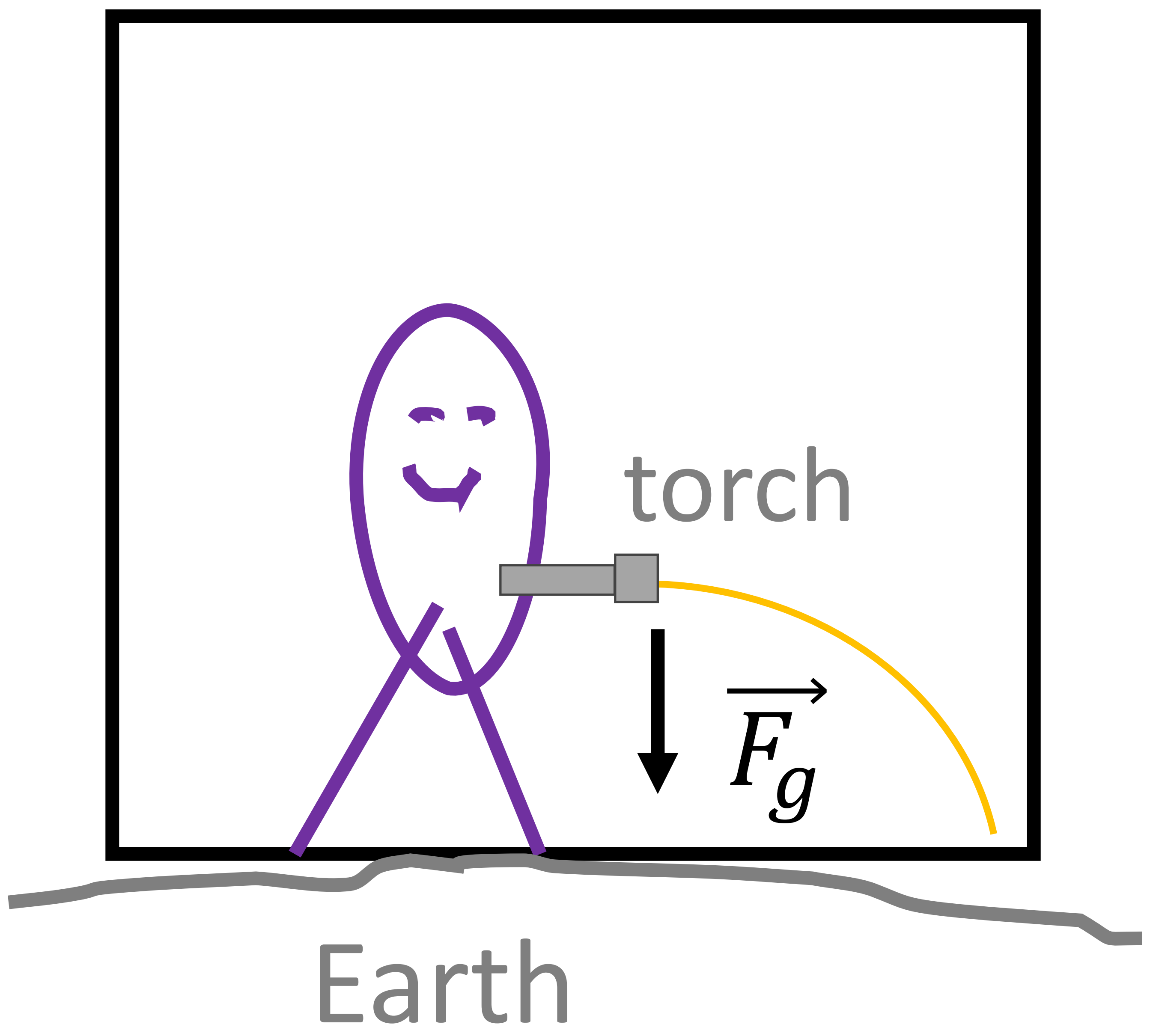
(2) Box accelerates & ball stays



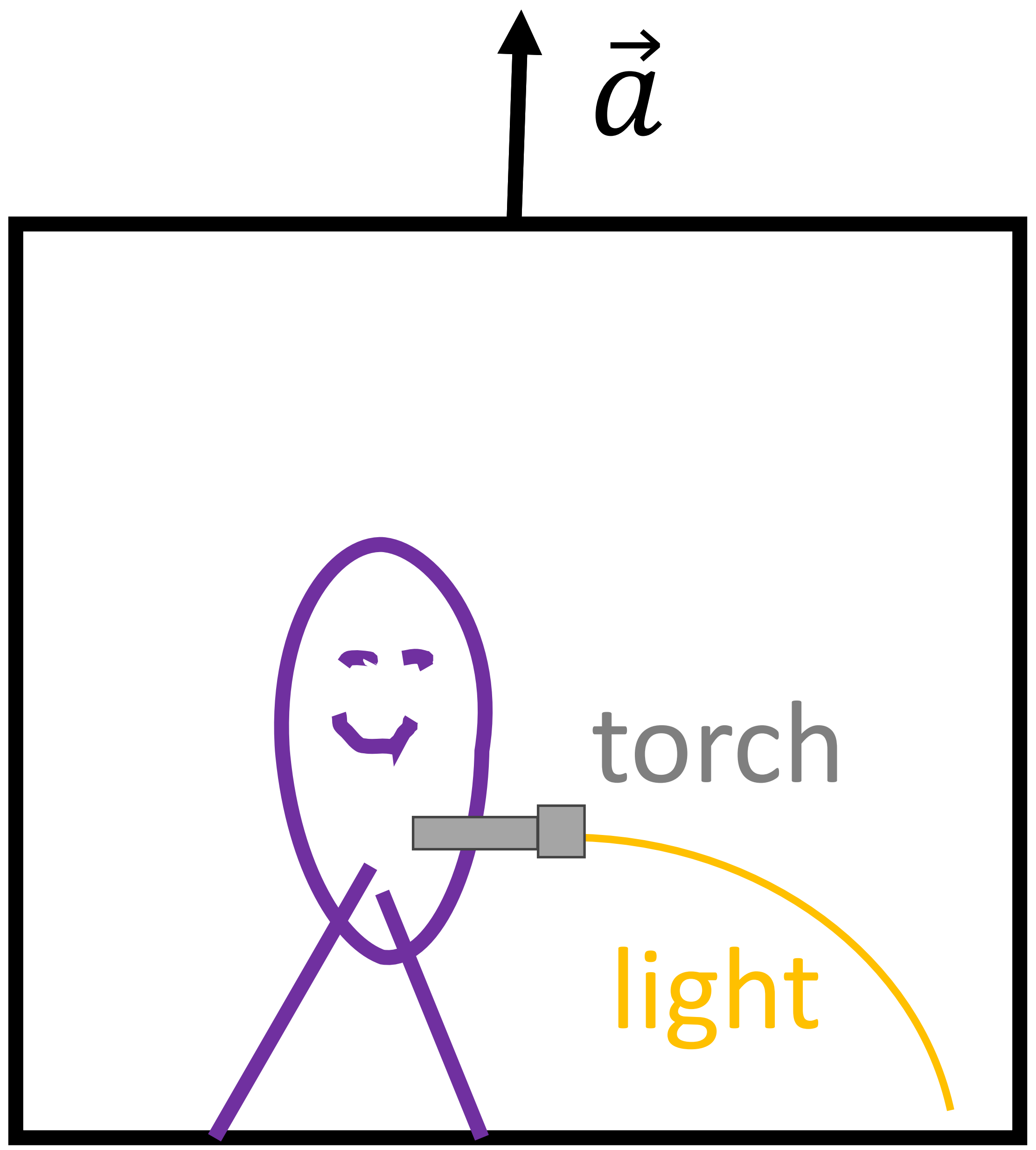
GENERAL RELATIVITY

- Same thought experiment with light: For the observer in a closed box the two situations are equal:

(1) Light bends under gravitation of Earth

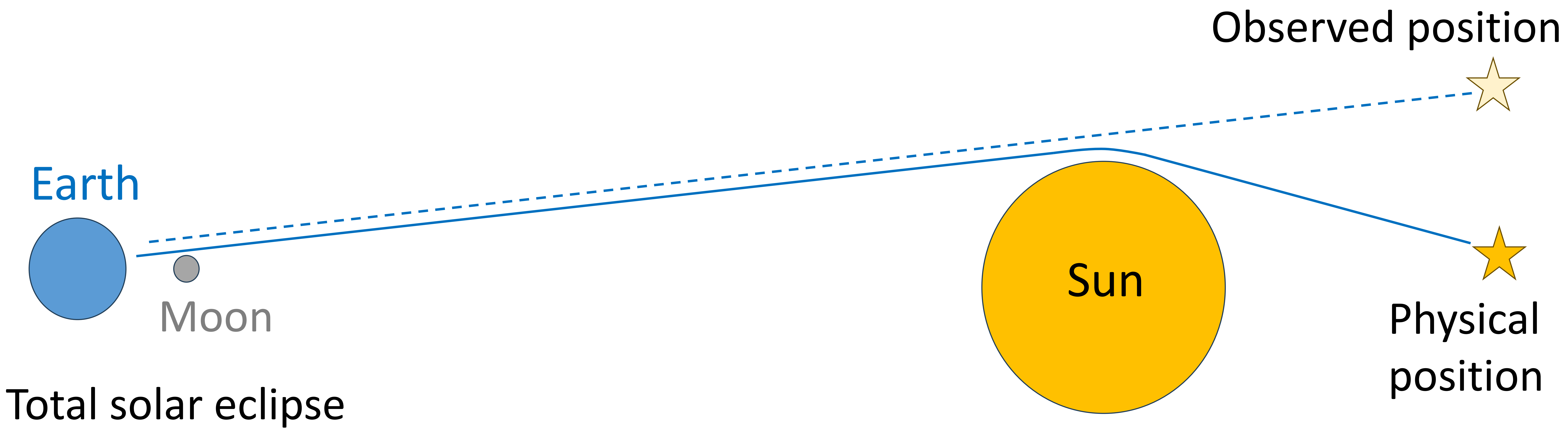


(2) Box accelerates & light bends



GENERAL RELATIVITY

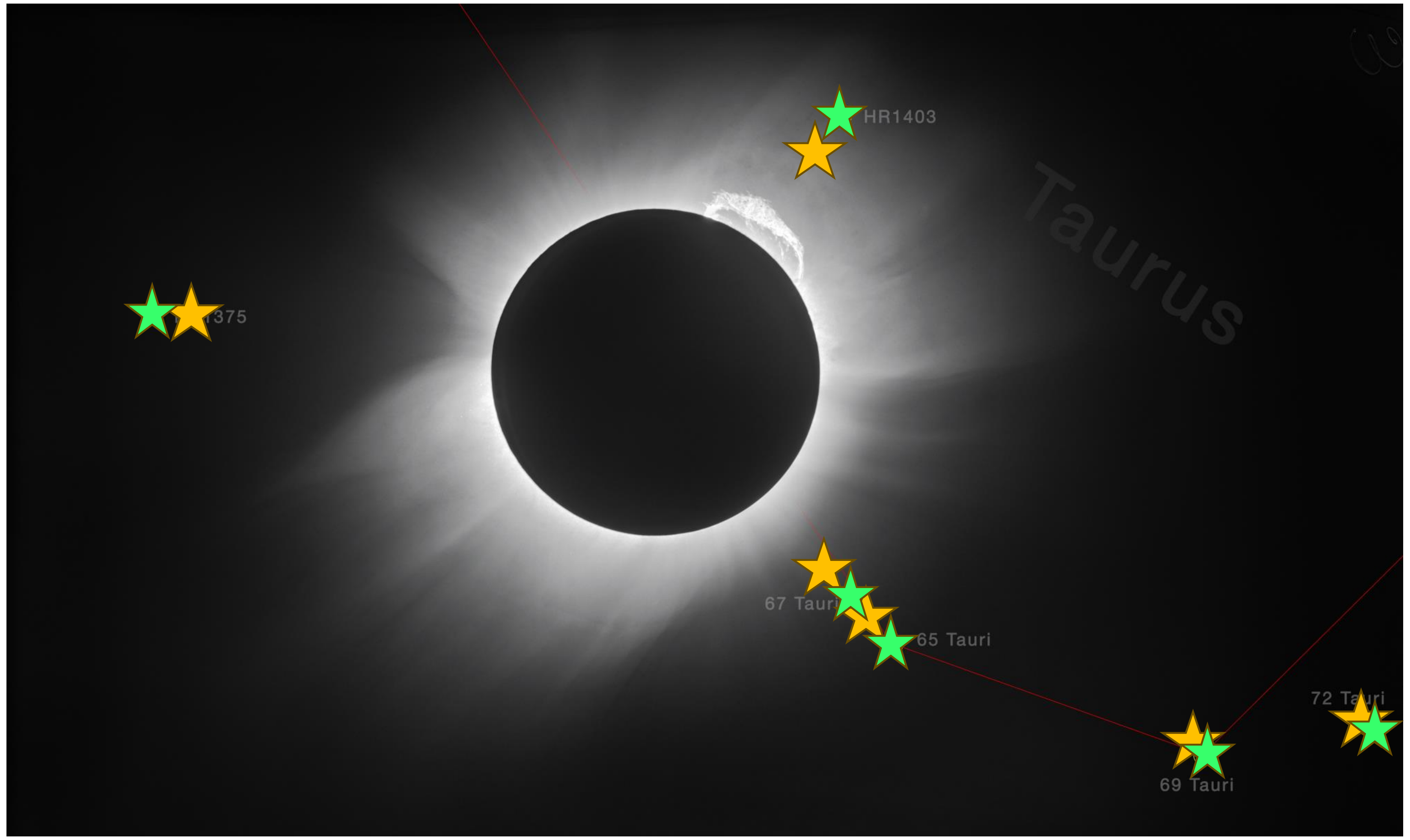
- Bending of light by mass was experimentally first in 1919 :
  - Eddington and Dyson took pictures of stars during a solar eclipse and at night (Sun does not bend the light)
  - Compared the positions of the stars with and “without” Sun



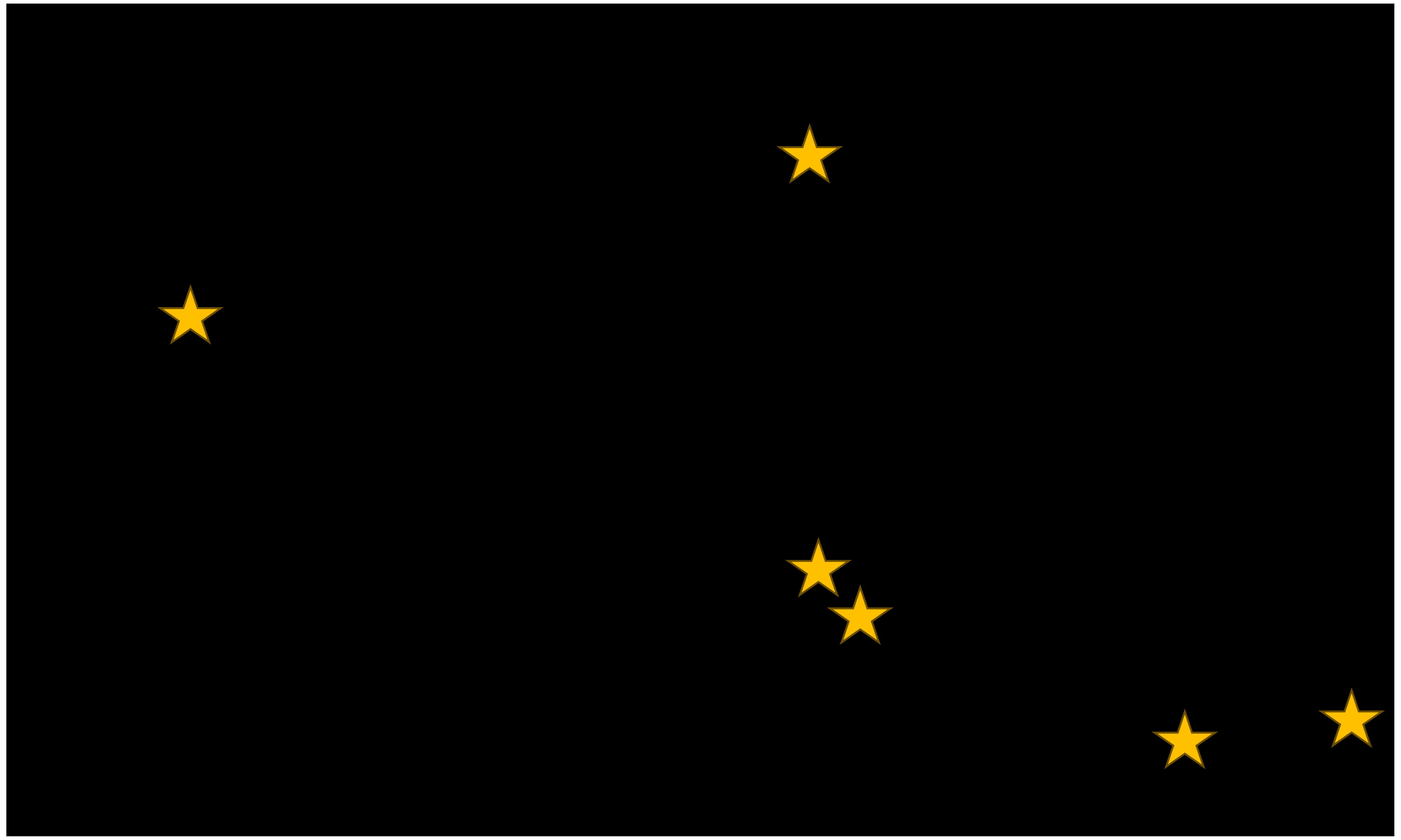
GENERAL RELATIVITY

- Bending of light by mass was experimentally first in 1919 :

Observed positions (in green) during the Sun eclipse: light bends



Physical positions (in orange) taking at night: light goes straight



GENERAL RELATIVITY: POSTULATES

Postulates of the general theory of relativity:

Principle of relativity: All laws of physics must be same in all inertial frames, **even if frame accelerates**

A gravitational field **is equivalent** to an accelerated frame in gravity-free space