PHOT 222: Quantum Photonics

Midterm exam 2: questions & solutions

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General information on the exam

Grading: This midterm exam will count for 20% of your total grade.

Exam type: The midterm exam consists of 4 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is given a score out of 5 points. For each question the sub-scores are then averaged.
- Answers should contain: The final formula/expression together with its derivation and a numerical approximate value with the **correct units**.

Question 1: Particle in a Box

Assume a particle that is confined within a 1D box of width L. According to de Broglie a quantum particle has a corresponding wave with wavelength $\lambda = h/p$ and energy $E = hf$.

(a) If the particle is a proton and the box has $L = 1$ nm, what is its minimum velocity according to de Broglie?
(b) If the "particle" is a football (weight: $m = 0.4$ kg) and the box has $L = 1$ m, what is its minimum velocity?

Solution (Q1)

(a) de Broglie tells us that due to the particle-wave duality we can assign a particle and therefore a velocity to the wave solutions of the particle in a box. The minimum velocity is the one corresponding to the lowest energy level.

$$E_1 = \frac{\hbar^2 \pi^2}{2mL^2} = \frac{1}{2} m v_1^2$$

The minimum velocity v_1 becomes:

$$\begin{aligned} v_1 &= \frac{\hbar \pi}{mL} = \frac{\pi \cdot 1.055 \times 10^{-34}\,\mathrm{J\,s}}{1836 \cdot 9.11 \times 10^{-31}\,\mathrm{kg}\,10^{-9}\,\mathrm{m}} \\ &= \frac{\pi \cdot 1.055}{1.836 \cdot 9.11} \times 10^3 \,\frac{\mathrm{kg\,m^2\,s^{-2}\,s}}{\mathrm{kg\,m}} \\ &\approx \frac{\pi}{18} \times 10^3\,\mathrm{m/s} \approx 1.5 \times 10^2\,\mathrm{m/s} \end{aligned}$$

An alternative way to get to the solution is from de Broglie relation $\lambda = h/p$ and knowledge of $\lambda_1 = 2L$ (particle in a box) for the lowest E_1:

$$p = \frac{h}{\lambda} \quad \Rightarrow \quad m v_1 = \frac{h}{2L} \Rightarrow v_1 = \frac{\hbar \pi}{mL}$$

leading to the same formula as before.

(b) When we change the proton for a football and the box to a one-meter large box, the minimum velocity becomes:

$$\begin{aligned} v_1 &= \frac{\hbar \pi}{mL} = \frac{\pi \cdot 1.055 \times 10^{-34}\,\mathrm{J\ s}}{0.4\,\mathrm{kg\,m}} \\ &= \frac{10\pi \cdot 1.055}{4} \times 10^{-34}\,\mathrm{m/s} \\ &\approx \frac{\pi}{4} \times 10^{-33}\,\mathrm{m/s} \end{aligned}$$

Question 2: Particle in a Box

An electron in a 1D infinitely deep square well of width L has wave function solutions $\psi_n(x)$ and E_n given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L}), \text{ with } x \in [0, L], \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

Assume that $L = 1$ nm and the electron is in the ground state $\psi_1(x)$, and gets excited to the $\psi_4(x)$ state by absorbing a photon.

(a) What is the energy required to excite the electron?
(b) What the corresponding wavelength of the photon to excite the electron?

Solution (Q2)

(a) The difference in energy $\Delta E = E_4 - E_1$ gives the required energy:

$$\begin{aligned}\Delta E = E_4 - E_1 &= \frac{\hbar^2\pi^2 4^2}{2mL^2} - \frac{\hbar^2\pi^2}{2mL^2} \\ &= \frac{\hbar^2\pi^2 4^2}{2mL^2}(4^2 - 1) = 15E_1 \\ &= 15\,\frac{\pi^2 \cdot (1.055)^2 \times 10^{-68}}{2 \cdot 9.11 \times 10^{-9}}\,\frac{\mathrm{J^2\,s^2}}{\mathrm{kg\,m^2}} \\ &\approx \frac{15\pi^2}{2 \cdot 9.11} \times 10^{-19}\,\mathrm{J}\end{aligned}$$

Then we convert to eV by using $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$:

$$\begin{aligned}\Delta E &\approx \frac{15\pi^2}{18} \times 10^{-19}\,\mathrm{J} \\ &\approx \frac{15 \cdot 9}{18 \cdot 1.6}\,\mathrm{eV} \approx 5\,\mathrm{eV}\end{aligned}$$

(b) The corresponding wavelength of the photon is given by the de Broglie energy $E = hf$:

$$\Delta E = hf = \frac{hc}{\lambda}$$

extracting the wavelength λ we obtain:

$$\lambda = \frac{hc}{\Delta E} \approx \frac{1240\,\mathrm{eV\ nm}}{5\,\mathrm{eV}} \approx 248\,\mathrm{nm}$$

Again a rough approximation due to the approximations made in the derivation of ΔE.

Question 3: Quantum Harmonic Oscillator

Consider a 1D quantum harmonic oscillator in the first excited state, which has wave function $\psi_1(x) = A_1 2\beta x e^{-\beta^2 x^2/2}$ where $\beta = \sqrt{\frac{m\omega}{\hbar}}$ and A_1 a normalization constant.

(a) Determine the value of the normalization constant A_1.
(b) Calculate the expectation value $\langle x^2 \rangle$. Simplify the resulting formula.

Solution (Q3)

(a) The normalization constant A_1 is obtained by putting the total probability to find the particle somewhere equal to one:

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} |\psi_1(x)|^2 \, dx = |A_1|^2 4\beta^2 \int_{-\infty}^{\infty} x^2 e^{-\beta^2 x^2} dx \\ &= |A_1|^2 4\beta^2 \frac{1}{2\beta^2} \frac{\sqrt{\pi}}{\beta} \\ &= |A_1|^2 \frac{2\sqrt{\pi}}{\beta}\end{aligned}$$

Then solving for A_1 filling in the expression for $\beta = \sqrt{\frac{m\omega}{\hbar}}$ and using $\hbar = h/2\pi$ we obtain:

$$|A_1|^2 = \frac{\sqrt{\frac{m\omega}{\hbar}}}{2\sqrt{\pi}} = \sqrt{\frac{m\omega}{2h}}$$

$$\Rightarrow \qquad A_1 = \left(\frac{m\omega}{2h}\right)^{1/4}$$

(b) For the expectation value $\langle x^2 \rangle$:

$$\begin{aligned}\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \, |\psi_1(x)|^2 \, dx = |A_1|^2 4\beta^2 \int_{-\infty}^{\infty} x^4 e^{-\beta^2 x^2} dx \\ &= |A_1|^2 4\beta^2 \frac{3}{4\beta^4} \frac{\sqrt{\pi}}{\beta} \\ &= |A_1|^2 \frac{3\sqrt{\pi}}{\beta^3} \\ &= \sqrt{\frac{m\omega}{2h}} 3 \sqrt{\frac{\pi\hbar}{m\omega}} \frac{\hbar}{m\omega} \\ &= \frac{3}{2} \frac{\hbar}{m\omega} = \frac{3}{2} \beta^{-2}\end{aligned}$$

Question 4: Tunneling probability

The tunneling probability T through a potential barrier for $T \ll 1$ can be approximated by following formula:

$$T \approx 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0}\right) e^{-2\kappa L}, \qquad \text{with} \quad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

Assume a quantum particle with energy $E = 1$ eV and that is incident on a 1D potential barrier with height $V_0 = 2$ eV, and width $L = 2$ nm.

(a) What is the tunneling probability T_a?
(b) If you change the width of the potential barrier to $L = 1$ nm, how much larger becomes the tunneling probability T_b? Give the formula for the ratio T_b/T_a and simplify as much as possible.

Solution (Q4)

(a) The transmission can be calculated with given approximate formula, let's first obtain κ:

$$\begin{aligned}\kappa &= \frac{\sqrt{2m(V_0-E)}}{\hbar} = \frac{\sqrt{2mc^2(V_0-E)}}{\hbar c} = \frac{\sqrt{2mc^2(V_0-E)}}{hc/(2\pi)} \\ &= \frac{\sqrt{2\cdot(0.511\,\mathrm{MeV})\cdot(1\,\mathrm{eV})\cdot 4\pi^2}}{1240\,\mathrm{eV\ nm}} \\ &= \frac{\sqrt{4\pi^2\cdot 2\cdot 0.511\times 10^6\,\mathrm{eV}^2}}{1240\,\mathrm{eV\ nm}} \\ &\approx \frac{\sqrt{4\pi^2\times 10^6}}{1.240\times 10^3}\mathrm{nm}^{-1} \approx \frac{4}{5}\sqrt{4\pi^2}\mathrm{nm}^{-1} \approx \frac{8\pi}{5}\,\mathrm{nm}^{-1} \approx 5\,\mathrm{nm}^{-1}\end{aligned}$$

Where we see that the exponent $-2\kappa L$ becomes dimensionless as it should (units in the exponent are problematic, try to imagine for example $2^{1\,\mathrm{m}} = 2^{10^9\,\mathrm{nm}}$, how would you calculate this?).

This gives for the transmission T:

$$T_a \approx 16\,\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)\,e^{-2\kappa L_a} = 16\frac{1}{4}\,e^{-\frac{32}{5}\pi} \approx 4\,e^{-20}$$

(b) If the length is half (i.e. $L_b = L_a/2$) then the exponential factor will become larger:

$$T_b \approx 16\,\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)\,e^{-2\kappa L_b} = 16\frac{1}{4}\,e^{-\frac{16\pi}{5}} \approx 4\,e^{-10}$$

The ratio becomes then (roughly):

$$\frac{T_b}{T_a} = \frac{e^{-2\kappa L_b}}{e^{-2\kappa L_a}} = e^{-2\kappa(L_b-L_a)} = e^{-2\cdot 5\,\mathrm{nm}^{-1}\,(-1\,\mathrm{nm})} \approx e^{10}$$

Values and formulas:

Mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of a proton: $m_p \approx 1836\ m_e$
1 eV $=$ 1.602 $\times 10^{-19}$ J
Joule in SI units: [J = kg m^2/s^2]
$h = 6.63 \times 10^{-34}$ J s $= 4.14 \times 10^{-15}$ eV s
$c = 3 \times 10^8$ m/s
$hc = 1240$ eV nm
$m_e c^2 = 0.511$ MeV

For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of a function $f(x)$ is:

$$\langle f(x) \rangle = \int_a^b \psi(x)^* \, f(x) \, \psi(x) dx.$$

Definite integrals:

$$\int_{-\infty}^{+\infty} x e^{-ax^2} \, dx = \frac{1}{a}$$
$$\int_{-\infty}^{+\infty} x^2 e^{-ax^2} \, dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$
$$\int_{-\infty}^{+\infty} x^3 e^{-ax^2} \, dx = \frac{1}{a^2}$$
$$\int_{-\infty}^{+\infty} x^4 e^{-ax^2} \, dx = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}}$$

Particle in a box: A quantum particle in a 1D infinitely deep square well of width L has wave function solutions $\psi_n(x)$ and E_n given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L}), \text{ with } x \in [0, L], \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$