PHOT 222: Quantum Photonics

Midterm exam 1: questions & solutions

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General information on the exam

Grading: This midterm exam will count for 20% of your total grade.

Exam type: The midterm exam consists of 4 open questions/problems. The exam is a written closed-book exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is given a score out of 5 points. For each question the sub-scores are then averaged.
- Answers should contain: The final formula/expression together with its derivation and a numerical approximate value with the **correct units**.

Question 1: Relativistic energie

An electron is traveling at a velocity $v = 0.6\,c$.

(a) What is the (relativistic) kinetic energy K of the electron?
(b) Then calculate the kinetic energy nonrelativistic ($K_{\text{classical}}$). What is the ratio between them:

$$\text{ratio} = \frac{K}{K_{\text{classical}}}$$

Solution (Q1)

(a) The relativistic kinetic energy is given by $K = (\gamma - 1)\, m_e c^2$ and γ is:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(3/5)^2 c^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{9}{25}}} = \frac{1}{\sqrt{\frac{16}{25}}} = \frac{5}{4}$$

Therefore K becomes:

$$K = (\gamma - 1)\, m_e c^2 = \left(\frac{5}{4} - 1\right)\, m_e c^2 = \frac{0.511}{4}\ \text{MeV} \approx 0.128\ \text{MeV}$$

(b) The nonrelativistic kinetic energy is given by:

$$K_{\text{classical}} = \frac{1}{2} m_e v^2 = \frac{1}{2}(3/5)^2 m_e c^2 = \frac{9}{50}\, 0.511\ \text{MeV} \approx 0.1\ \text{MeV}$$

And the ratio becomes:

$$\text{ratio} = \frac{K}{K_{\text{classical}}} = \frac{\frac{1}{4} m_e c^2}{\frac{9}{50} m_e c^2} = \frac{25}{18}$$

Question 2: Photoelectric effect

When we shine UV light onto a carbon (C) plate photoelectrons are emitted from the surface. Assume that carbon has a workfunction $\phi = 5.0$ eV.

(a) What is the minimum frequency of the light to have photoemission for this carbon surface?
(b) If we use UV light with a wavelength of 100 nm, what is the maximum velocity v_{max} of the electrons?

Solution (Q2)

(a) The minimum frequency is at condition $hf = \phi$, therefore:

$$f = \frac{\phi}{h} \approx \frac{5\ \text{eV}}{4.14 \times 10^{-15}\ \text{eV s}} = \frac{5}{4.14} \times 10^{15}\ \text{Hz} \approx 1.2 \times 10^{15}\ \text{Hz}$$

(b) The maximum velocity v_{max} of the photoelectrons will be at maximum kinetic energy $K_{\text{max}} = \frac{1}{2} m_e v_{\text{max}}^2$:

$$K_{\max} = hf - \phi = \frac{hc}{\lambda} - \phi = \frac{1240 \text{ eV nm}}{100 \text{ nm}} - 5 \text{ eV} = (12.4 - 5) \text{ eV} = 7.4 \text{ eV}$$

$$\Rightarrow \quad v_{\max} = \sqrt{\frac{2K_{\max}}{m_e}} = \sqrt{\frac{14.8 \text{ eV}}{m_e c^2}}\, c = \sqrt{\frac{14.8 \text{ eV}}{0.511 \times 10^6 \text{ eV}}}\, c = \sqrt{\frac{14.8}{0.511}} \times 10^{-3}\, c \approx 1.6 \times 10^6 \text{ m/s}$$

Which is sufficiently small for the nonrelativistic approximation of the kinetic energy.

Question 3: de Broglie

A proton travels at $v_p = 0.01c$ with c the speed of light.

(a) What is the de Broglie wavelength $\lambda = h/p$ for this proton?
(b) An electron with the same de Broglie wavelength λ would go faster, what is its approximate velocity $v_e = ?$ (Use a relativistic description for the electron).
(c) If you would calculate v_e in a nonrelativistic manner, what would be the result? Is that faster or slower than the speed of light?

Solution (Q3)

(a) The de Broglie wavelength is given by:

$$\lambda = \frac{h}{p} = \frac{h}{m_p v} = \frac{hc}{m_p 0.01 c^2} = \frac{1240 \text{ eV nm}}{0.01 \cdot 1836 \cdot m_e c^2} \approx \frac{1240 \text{ eV nm}}{18.36 \cdot 0.511 \times 10^6 \text{eV}} \approx \frac{4}{3} \times 10^{-11} \text{ m}$$

(b) The condition for having the same de Broglie wavelength $\frac{h}{p_e} = \frac{h}{p_p}$ results in:

$$p_e = p_p$$
$$\Rightarrow \quad \gamma_e m_e v_e = m_p v_p$$

Where we use the relativistic description of the momentum $\gamma m v$ for the electron (the proton is slow enough so we can use the classical approximation there). In following derivation we denote $\beta \equiv v_e/c$ and divide the equation by c:

$$\Rightarrow \quad \frac{1}{\sqrt{1-\frac{v_e^2}{c^2}}} m_e v_e/c = m_p v_p/c$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-\beta^2}} m_e \beta = m_p v_p/c$$

$$\Rightarrow \quad \frac{1}{1-\beta^2} m_e^2 \beta^2 = m_p^2 v_p^2/c^2$$

$$\Rightarrow \quad \frac{m_e^2 c^2}{m_p^2 v_p^2} = \frac{1}{\beta^2} - 1$$

$$\Rightarrow \quad \frac{c^2}{(1836)^2 \; v_p^2} + 1 = \frac{c^2}{v_e^2}$$

$$\Rightarrow \quad v_e = \sqrt{\frac{1}{\frac{c^2}{(1836)^2 \, 0.01^2 \, c^2} + 1}}\, c = \sqrt{\frac{1}{\frac{1+(18.36)^2}{(18.36)^2}}}\, c = \sqrt{\frac{(18.36)^2}{1+(18.36)^2}}\, c$$

Question 4: Wave functions and probability

Consider the following wave function defined with $x \in \; [0, \infty[$.

$$\psi(x) = A\, x\, e^{-x/2},$$

with A a normalization constant.

(a) Calculate the normalization constant A of the wave function.
(b) Determine the probability for the particle to be found within interval $[0, 1]$. The Euler constant $e \approx 2.718$, but you can write the end-result as a formula.
(c) Afterwards calculate the expectation value for the position x, that is, calculate $\langle x \rangle$.

Solution (Q3)

(a) The normalization constant A is obtained by putting the total probability equal to one:

$$1 = \int_0^\infty |\psi(x)|^2 \, dx = |A|^2 \int_0^\infty x^2 e^{-x} \, dx = |A|^2 \left[-(x^2 + 2x + 2)\, e^{-x} \right] \Big|_0^\infty = |A|^2[-0-(-2)] = 2|A|^2$$

$$|A|^2 = \frac{1}{2} \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}}$$

(b) The probability to find the particle in $[0, 1]$ is:

$$
\begin{aligned}
P(x \in [0,1]) &= \int_0^1 |\psi(x)|^2 \, dx \\
&= |A|^2 \int_0^1 x^2 e^{-x} \, dx \\
&= \frac{1}{2} \left[-(x^2 + 2x + 2) \, e^{-x} \right] \Big|_0^1 \\
&= \frac{1}{2} \left[-\frac{5}{e} + 2 \right] \\
&= \frac{e - 2.5}{e}
\end{aligned}
$$

(c) The expectation value $\langle x \rangle$ is given by:

$$
\begin{aligned}
\langle x \rangle = \int_0^\infty x \, |\psi(x)|^2 \, dx = |A|^2 \int_0^\infty x^3 e^{-x} \, dx \\
&= \frac{1}{2} \left[-(x^3 + 3x^2 + 6x + 6) \, e^{-x} \right] \Big|_0^\infty \\
&= \frac{1}{2} \left(-0 - (-6) \right) = 3
\end{aligned}
$$

Values and formulas:

Mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of a proton: $m_p \approx 1836\ m_e$
1 eV $= 1.602 \times 10^{-19}$ J
A Joule has units of: [J = kg m^2/s^2]
$h = 6.63 \times 10^{-34}$ J s $= 4.14 \times 10^{-15}$ eV s
$c = 3 \times 10^8$ m/s
$hc = 1240$ eV nm
$m_e c^2 = 0.511$ MeV

For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of a function $f(x)$ is:

$$\langle f(x) \rangle = \int_a^b f(x)\, |\psi(x)|^2\, dx.$$

You can also make use of following indefinite integrals (anti-derivatives):

$$\int x e^{-x}\, dx = -(x+1)\, e^{-x}$$

$$\int x^2 e^{-x}\, dx = -(x^2 + 2x + 2)\, e^{-x}$$

$$\int x^3 e^{-x}\, dx = -(x^3 + 3x^2 + 6x + 6)\, e^{-x}$$