PHOT 222: Quantum Photonics

Midterm exam 1: example questions & solutions

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General information on the exam

Grading: This midterm exam will count for 20% of your total grade.

Exam type: The midterm exam consists of 4 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions & solutions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is given a score out of 5 points. For each question the sub-scores are then averaged.
- Answers should contain: The final formula/expression together with its derivation and a numerical approximate value with the **correct units**.

Question 1: Relativistic energie

(a) What is the rest energy of a photon?
(b) What is the total energy of an electron traveling at $v = 0.8c$. Assume that the electron has a mass m_e.
(c) Consider an electron with mass m_e travels at velocity v and a proton that is at rest. What is the velocity v of the electron when the total energy of both particles is equal?

Solution (Q1)

(a) The rest energy $E_R = mc^2$ of a photon is zero as it has no mass

(b) The total energy of an electron with velocity $0.8c = \frac{4}{5}c$:

$$E = \gamma m_e c^2 = \frac{1}{\sqrt{1 - \frac{(4/5)^2 c^2}{c^2}}} m_e c^2 = \frac{1}{\sqrt{1 - \frac{16}{25}}} m_e c^2 = \frac{5}{3} m_e c^2 = \frac{5}{3} 5.11 \text{ MeV} \approx 8.5 \text{ MeV}$$

(c) The proton has total energy equal to its rest energy $E_p = m_p c^2$. The total energy of the electron is $E_e = \gamma m_e c^2$:

$$m_p c^2 = \gamma m_e c^2 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} m_e c^2$$

$$\Rightarrow \quad \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_e c^2}{m_p c^2}$$

$$\Rightarrow \quad 1 - \frac{v^2}{c^2} = \frac{m_e^2}{m_p^2}$$

$$\Rightarrow \quad \frac{v^2}{c^2} = 1 - \frac{m_e^2}{m_p^2}$$

$$\Rightarrow \quad v = \sqrt{1 - \frac{m_e^2}{m_p^2}}\, c = \sqrt{1 - \frac{1}{1836^2}}\, c \approx \left(1 - \frac{1}{2 \cdot 1836^2}\right) 3 \times 10^8 \text{ m/s}$$

Where we used the approximation $\sqrt{1-x} \approx 1 - x/2$, and this last approximation is not obligatory for the numerical value.

Question 2: Waves and particles

According to de Broglie particles having momentum p have a corresponding wavelength $\lambda = h/p$.

(a) What is the wavelength λ for an electron traveling at a velocity $v = 1000$ m/s?
(b) How fast should a proton approximately go to obtain the same corresponding wavelength λ of the electron traveling at a velocity $v = 1000$ m/s?

Solution (Q2)

(a) The de Broglie wavelength can be calculated via $\lambda = \frac{h}{p}$ with $p = m_e v$ nonrelativistic because $v = 1000$ m/s $\ll c$:

$$\lambda = \frac{h}{p} = \frac{h}{m_e v} = \frac{6.63 \times 10^{-34} \text{ J s}}{9.11 \times 10^{-31} \text{ kg} \cdot 1000 \text{ m/s}} = \frac{6.63}{9.11} \times 10^{-6} \text{ m} \approx 0.7 \text{ um}$$

where the last approximation is very crude.

(b) If the proton has the same wavelength, i.e. $\lambda_p = \lambda_e$ (where the index p stands for proton, and e stands for electron):

$$\frac{h}{p_p} = \lambda_p = \lambda_e = \frac{h}{p_e}$$

$$\Rightarrow \quad m_p v_p = m_e v_e$$

$$\Rightarrow \quad v_p = \frac{m_e v_e}{m_p} \approx \frac{m_e v_e}{1836 \cdot m_e} = \frac{v_e}{1836} = \frac{1000}{1836} \text{ m/s}$$

Remark that in the above derivation we could use the nonrelativistic formula for the momenta of both electron and proton because their velocities are much less than the speed of light.

Question 3: Photoelectric effect

When we shine UV light onto a silver (Ag) plate photoelectrons are emitted from the surface. Assume that Silver has a workfunction $\phi = 4.3$ eV.

(a) If we use UV light with a wavelength of 200 nm, what is the maximum kinetic energy K_{max} of the electrons?
(b) What is the cutoff wavelength to have photoemission for a silver plate?

Solution (Q3)

(a) The maximum kinetic energy can be derived as follows K_{max}

$$K_{\text{max}} = hf - \phi = \frac{hc}{\lambda} - \phi = \frac{1240 \text{ eV nm}}{200 \text{ nm}} - 4.3 \text{ eV} = (5.2 - 4.3) \text{ eV} = 0.9 \text{ eV}$$

(b) The cutoff wavelength λ_c has condition $hf = \phi$:

$hf = \frac{hc}{\lambda_c} = \phi \Rightarrow \lambda_c = \frac{hc}{\phi} = \frac{1240 \text{ eV nm}}{4.3 \text{ eV}} \approx 290$ nm

In this approximation the last decimal is not accurate.

Question 4: Wave functions and probability

Consider the following wave function defined with $x \in \mathbb{R}$ and $a > 0$:

$$\psi(x) = A \frac{1}{\sqrt{x^2 + a^2}},$$

with A a normalization constant.

(a) Calculate the normalization constant A of the wave function.
(b) Determine the probability for the particle to be found within interval $[0, a]$.
(c) Afterwards calculate the expectation value for the function $f(x) = x$, that is, calculate $\langle x \rangle$.

Solution (Q4)

(a) The normalization coefficients is derived by putting the probability equal to one:

$$\begin{aligned}
1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = |A|^2 \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \, dx \\
&= |A|^2 \frac{1}{a} \arctan\left(\frac{x}{a}\right) \Big|_{-\infty}^{+\infty} \\
&= |A|^2 \frac{1}{a} \left(\arctan\left(\frac{\infty}{a}\right) - \arctan\left(-\frac{\infty}{a}\right) \right) \\
&= |A|^2 \frac{1}{a} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) \\
&= |A|^2 \frac{\pi}{a} \\
\Rightarrow A = \sqrt{\frac{a}{\pi}}
\end{aligned}$$

(b) The probability to be found in interval $x \in [0, a]$:

$$\begin{aligned}
P(x \in [0, a]) = \int_{0}^{a} |\psi(x)|^2 \, dx = |A|^2 \int_{0}^{a} \frac{1}{x^2 + a^2} \, dx \\
&= |A|^2 \frac{1}{a} \arctan\left(\frac{x}{a}\right) \Big|_{0}^{a} \\
&= |A|^2 \frac{1}{a} \left(\arctan\left(\frac{a}{a}\right) - \arctan(0) \right) \\
&= \frac{a}{\pi} \frac{1}{a} (1 - 0) = \frac{1}{\pi}
\end{aligned}$$

(c) Expectation value $\langle x \rangle$ is derived as follows:

$$
\begin{aligned}
\langle x\rangle = \int_{-\infty}^{\infty} x\,|\psi(x)|^2\,dx &= |A|^2 \int_{-\infty}^{\infty} \frac{x}{x^2+a^2}\,dx \\
&= |A|^2 \frac{1}{2}\,\log\left(x^2+a^2\right)\Big|_{-\infty}^{+\infty} \\
&= |A|^2 \frac{1}{2}\,\log\left(x^2+a^2\right)\Big|_{-\infty}^{+\infty} \\
&= \frac{a}{2\pi}\lim_{x\to\infty}\left[\log\left(x^2+a^2\right)-\log\left((-x)^2+a^2\right)\right] \\
&= \frac{a}{2\pi}\lim_{x\to\infty}\left[\log\left(\frac{x^2+a^2}{x^2+a^2}\right)\right] = \frac{a}{2\pi}
\end{aligned}
$$

Values and formulas:

Mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of a proton: $m_p \approx 1836\ m_e$
1 eV $= 1.602 \times 10^{-19}$ J
A Joule has units of: [J = kg m^2/s^2]
$h = 6.63 \times 10^{-34}$ J s $= 4.14 \times 10^{-15}$ eV s
$c = 3 \times 10^8$ m/s
$hc = 1240$ eV nm
$m_e c^2 = 0.511$ MeV

For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of a function $f(x)$ is:

$$\langle f(x) \rangle = \int_a^b f(x)\,|\psi(x)|^2\,dx.$$

You can also make use of following indefinite integrals (anti-derivatives):

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \frac{1}{2}\left[\log\left(1+\frac{x}{\sqrt{x^2+a^2}}\right) + \log\left(1-\frac{x}{\sqrt{x^2+a^2}}\right)\right]$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\ \arctan\left(\frac{x}{a}\right)$$

$$\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2}$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ \log\left(x^2+a^2\right)$$