PHOT 222: Quantum Photonics

Midterm exam 1C: questions & solutions

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General information on the exam

Grading: This midterm exam will count for 20% of your total grade.

Exam type: The midterm exam consists of 4 open questions/problems. The exam is a written closed-book exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is given a score out of 5 points. For each question the sub-scores are then averaged.
- Answers should contain: The final formula/expression together with its derivation and a numerical approximate value with the **correct units**.

Question 1: Relativistic energie

An electron is accelerated from stillstand to a velocity of $v_1 = 0.6\,c$. Afterwards it is further accelerated to $v_2 = 0.8\,c$, where c is the speed of light.

(a) What is its (relativistic) momentum at velocity v_1?
(b) What is its (relativistic) kinetic energy K_1 at velocity v_1?
(c) What is the kinetic energy K_2 after the second acceleration? And what is the extra energy required to reach from v_1 to v_2?

Solution (Q1)

(a) The relativistic momentum p_1 at velocity v_1 is given by $\gamma_1 m_e v_1$:

$$\gamma_1 m_e v_1 = \frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} m_e v_1 = \frac{1}{\sqrt{1-0.6^2}} m_e \, 0.6 \, c = \frac{0.6}{0.8} m_e c$$
$$= \frac{3}{4} (9.11 \times 10^{-31}) \, (3 \times 10^8) \text{ kg m/s} \approx 2.05 \times 10^{-22} \text{ kg m/s}$$

(b) The relativistic kinetic energy K_1 is given by:

$$(\gamma - 1) m_e c^2 = \left(\frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} - 1 \right) m_e c^2 = \left(\frac{5}{4} - 1 \right) m_e c^2 = \frac{1}{4} 0.511 \text{ MeV} \approx 0.128 \text{ MeV}$$

(c) At velocity v_2 the relativistic kinetic energy K_2 is given by:

$$(\gamma - 1) m_e c^2 = \left(\frac{1}{\sqrt{1-\frac{v_2^2}{c^2}}} - 1 \right) m_e c^2 = \left(\frac{5}{3} - 1 \right) m_e c^2 = \frac{2}{3} 0.511 \text{ MeV} \approx 0.33 \text{ MeV}$$

Therefore the extra energy required is:

$$K_2 - K_1 = \left(\frac{2}{3} - \frac{1}{4} \right) 0.511 \text{ MeV} = \frac{5}{12} 0.511 \text{ MeV} \approx 0.21 \text{ MeV}$$

Question 2: Blackbody radiation

In blackbody radiation, Wien's law of displacement describes how the peak in the wavelength distribution shifts towards shorter wavelengths for increasing temperature T, and is given by the following formula:

$$\lambda_{\text{max}} T = 2.898 \times 10^{-3} \, \text{m K},$$

where λ_{max} is the peak wavelength.

(a) Calculate the temperature T for λ_{max} at 500 nm.
(b) What is (approximately) the peak wavelength λ_{max} for a black body with a surface temperature $T = 20^\circ\text{C} = 293$ K?

Solution (Q2)

(a) T for λ_{max} at 500 nm is given by:

$$T = \frac{2.898 \times 10^{-3}\,\text{m K}}{\lambda_{\text{max}}} = \frac{2.898 \times 10^{-3}\,\text{m K}}{0.5 \times 10^{-6}\ \text{m}} = 5796\ \text{K}$$

(b) The peak wavelength is given by:

$$\lambda_{\text{max}} = \frac{2.898 \times 10^{-3}\,\text{m K}}{T} = \frac{2.898 \times 10^{-3}\,\text{m K}}{293\ \text{K}} \approx \frac{2.898}{2.93} \times 10^{-5}\ \text{m} \approx 10\ \text{um}$$

Question 3: de Broglie

(a) What is the momentum p of a photon that has a wavelength $\lambda = 600$ nm.
(b) If an electron has the same momentum p, what is its velocity (in m/s)? Did you need to use the relativistic expression for momentum?

Solution (Q3)

(a) The momentum is derived from $\lambda = \frac{h}{p}$:

$$p = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34}\ \text{J s}}{600 \times 10^{-9}\ \text{m}} \approx 1.1 \times 10^{-25}\ \text{kg m/s}$$

(b) We try to use first the classical definition of the momentum $p = p_e = m_e v_e$:

$$v_e = \frac{p}{m_e} = \frac{1.1 \times 10^{-25}\ \text{kg m/s}}{9.11 \times 10^{-31}\ \text{kg}} = \frac{1.1}{9.11} \times 10^{6}\ \text{m/s} \approx 12 \times 10^{5}\ \text{m/s}$$

Since the resulting velocity v_e is much smaller than c the nonrelativistic approximation is appropriate.

Question 4: Wave functions and probability

Consider the following wave function $\psi(x)$ defined with $x \in [0, \pi/2]$:

$$\begin{aligned} \psi(x) = A\,\sin(x)\,\cos(x), &\qquad \text{if } x \in [0, \pi/2], \\ \psi(x) = 0, &\qquad \text{if } x > \pi/2 \text{ or } x < 0, \end{aligned}$$

with A a normalization constant.

(a) Calculate the normalization constant A of the wave function.
(b) What is the probability to find the particle in interval $[0, \pi/4]$?
(c) Afterwards calculate the expectation value for $f(x) = x$, that is, calculate $\langle x \rangle$.

Solution (Q4)

(a) The normalization constant is found by putting the total probability equal to one:

$$
\begin{aligned}
1 = \int_0^{\pi/2} |\psi(x)|^2 \, dx = |A|^2 \int_0^{\pi/2} \sin^2 x \, \cos^2 x \, dx \\
&= |A|^2 \frac{1}{32} \left(4x - \sin(4x)\right) \Big|_0^{\pi/2} \\
&= \frac{|A|^2}{30} |A|^2 \frac{1}{32} \left((2\pi - \sin(2\pi)) - 0\right) \\
&= |A|^2 \frac{\pi}{16}
\end{aligned}
$$

$$
|A|^2 = \frac{16}{\pi} \quad \Rightarrow \quad A = \frac{4}{\sqrt{\pi}}
$$

(b) The probability to find the particle in $[0, \pi/4]$ is given by:

$$
\begin{aligned}
P(x \in [0, \pi/4]) &= \int_0^{\pi/4} |\psi(x)|^2 \, dx \\
&= |A|^2 \int_0^{\pi/4} \sin^2 x \, \cos^2 x \, dx \\
&= |A|^2 \frac{1}{32} \left(4x - \sin(4x)\right) \Big|_0^{\pi/4} \\
&= |A|^2 \frac{1}{32} (\pi - 0) \\
&= \frac{16}{\pi} \frac{1}{32} \pi \\
&= \frac{1}{2}
\end{aligned}
$$

This could also be argued from symmetry of the probability density around $\pi/4$.

(c) The expectation value $\langle x \rangle$ is given by:

$$
\begin{aligned}
\langle x \rangle = \int_0^{\pi/2} x \, |\psi(x)|^2 \, dx = |A|^2 \int_0^{\pi/2} x \, \sin^2 x \, \cos^2 x \, dx \\
&= |A|^2 \frac{1}{128} \left(8x^2 - 4x \sin(4x) - \cos(4x)\right) \Big|_0^{\pi/2} \\
&= \frac{16}{\pi} \frac{1}{128} \left((2\pi^2 - 1) - (-1)\right) \\
&= \frac{\pi}{4}
\end{aligned}
$$

Again, this could be argued from symmetry of the probability density function as well.

Values and formulas:

Mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of a proton: $m_p \approx 1836\ m_e$
1 eV $=$ 1.602 $\times$ 10^{-19} J
Joule in SI units: [J = kg m^2/s^2]
$h = 6.63 \times 10^{-34}$ J s $= 4.14 \times 10^{-15}$ eV s
$c = 3 \times 10^8$ m/s
$hc = 1240$ eV nm
$m_e c^2 = 0.511$ MeV

For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of a function $f(x)$ is:

$$\langle f(x) \rangle = \int_a^b f(x)\, |\psi(x)|^2\, dx.$$

You can also make use of following indefinite integrals:

$$\int x\ \cos(x)\, dx = x\ \sin(x) + \cos(x)$$

$$\int \sin(x)\cos(x) dx = -\frac{1}{2}\cos^2(x)$$

$$\int x\ \sin(x)\cos(x) dx = \frac{1}{8}\left(\sin(2x) - 2x\ \cos(2x)\right)$$

$$\int \sin^2(x)\cos^2(x) dx = \frac{1}{32}\left(4x - \sin(4x)\right)$$

$$\int x \sin^2(x)\cos^2(x) dx = \frac{1}{128}\left(8x^2 - 4x\sin(4x) - \cos(4x)\right)$$