PHOT 222: Quantum Photonics

Midterm exam 1B: questions & solutions

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General information on the exam

Grading: This midterm exam will count for 20% of your total grade.

Exam type: The midterm exam consists of 4 open questions/problems. The exam is a written closed-book exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the midterm exam is 2 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

This document contains both the problems and their solutions. Considering the scoring calculation:

- when you have to answer multiple subproblems each of the subtasks is given a score out of 5 points. For each question the sub-scores are then averaged.
- Answers should contain: The final formula/expression together with its derivation and a numerical approximate value with the **correct units**.

Question 1: Relativistic energie

A proton has a total energy E of two times its kinetic energy K.

(a) What is the (relativistic) kinetic energy K of the proton?
(b) What is the corresponding velocity of the proton?

Solution (Q1)

The total energy is defined as the sum of rest energy E_R and kinetic energy K:

$$E = K + E_R = (\gamma - 1)m_p c^2 + m_p c^2$$

(a) If total energy of the proton $E = K + E_R = 2K$ then:

$$K = E_R = m_p c^2 = 1836\, m_e c^2 = 1836 \cdot 0.511 \text{ MeV} \approx 3.7 \text{ GeV}$$

(a) The velocity of the proton can be derived from the fact that $\gamma = 2$ (otherwise $E_R \neq K$):

$$2 = \gamma = \frac{1}{\sqrt{1 - \frac{v_e^2}{c^2}}} \Rightarrow \frac{1}{1 - \frac{v_e^2}{c^2}} = 4 \Rightarrow 1 - \frac{v_e^2}{c^2} = \frac{1}{4} \Rightarrow v_e = \sqrt{\frac{3}{4}}\, c$$

Question 2: The Compton shift

A photon with wavelength $\lambda = 0.1nm$ hits a standing still free electron during a Compton measurement. The photon scatters under an angle $\theta = 60^\circ$. The Compton shift is given by the following formula:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos(\theta))$$

(a) What is wavelength λ' of the photon after the collision?

Solution (Q2)

(a) Use the fact that $\cos(60^\circ) = \frac{1}{2}$

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos(\theta)) = \lambda + \frac{h}{2m_e c} = \lambda + \frac{hc}{2m_e c^2}$$

Fill in the known values:

$$\lambda' = 0.1 \text{ nm} + \frac{1240 \text{ eV nm}}{2 \cdot 0.511 \times 10^6 \text{ eV}} \approx 0.1 \text{ nm} + 1.240 \times 10^{-3} \text{ nm} = 0.1012 \text{ nm}$$

Question 3: de Broglie

(a) A photon has energy $E = 5$ eV. What is its wavelength λ?
(b) What is the de Broglie wavelength of an electron that has the same kinetic energy.
(c) What is the velocity of the electron?

Solution (Q3)

(a) For a photon the energy $E = hf = \frac{hc}{\lambda}$, therefore the wavelength is given by:

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV nm}}{5.0 \text{ eV}} = 248 \text{ nm}$$

(b) The de Broglie wavelength of an electron is given by $\lambda = \frac{h}{p}$, where the momentum $p = m_e v_e$ can be derived from $K = \frac{1}{2} m_e v_e^2$. We further can use the nonrelativistic approximations of the momentum/energy since 5 eV $\ll m_e c^2 = 0.511$ MeV:

$$v_e = \sqrt{\frac{2K}{m_e}} = \sqrt{\frac{2K}{m_e c^2}}\, c$$

Therefore the de Broglie wavelength is:

$$\lambda = \frac{h}{p} = \frac{h}{m_e c \sqrt{\frac{2K}{m_e c^2}}} = \frac{hc}{\sqrt{2K m_e c^2}} = \frac{1240 \text{ eV nm}}{\sqrt{10.0 \text{ eV} \cdot 0.511 \times 10^6 \text{ eV}}} \approx \frac{1240}{\sqrt{20}} \times 10^{-3} \text{ nm} \approx \frac{1.240}{4.5} \text{ nm}$$

(c): The velocity is as calculated in (b) above:

$$v_e = \sqrt{\frac{2K}{m_e}} = \sqrt{\frac{2K}{m_e c^2}}\, c = \sqrt{10.0 \text{ eV} 0.511 \times 10^6 \text{ eV}}\, c \approx \sqrt{20.0} \times 10^{-3}\, c \approx 13.5 \times 10^5 \text{ m/s}$$

Question 4: Wave functions and probability

Consider the following wave function defined with $x \in [0, 1]$:

$$\psi(x) = A\, x\,(1 - x), \qquad \text{If } x \in [0, 1]$$
$$\psi(x) = 0, \qquad \text{If } x > 1 \text{ or } x < 0$$

with A a normalization constant.

(a) Calculate the normalization constant A of the wave function.
(b) Afterwards calculate the expectation value for $f(x) = x^2$, that is, calculate $\langle x^2 \rangle$.

Solution (Q4)

(a) The normalization constant is found by putting the total probability equal to one:

$$1 = \int_0^1 |\psi(x)|^2 \, dx = |A|^2 \int_0^1 x^2 \, (1-x)^2 \, dx = |A|^2 \frac{2!2!}{5!} = \frac{|A|^2}{30}$$

$$|A|^2 = 30 \quad \Rightarrow \quad A = \sqrt{30}$$

(b) The expectation value $\langle x^2 \rangle$ is given by:

$$\langle x^2 \rangle = \int_0^1 x^2 \, |\psi(x)|^2 \, dx = |A|^2 \int_0^1 x^4 (1-x)^2 \, dx$$
$$= 30 \, \frac{4!2!}{7!} = \frac{2}{7}$$

Values and formulas:

Mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of a proton: $m_p \approx 1836\ m_e$
1 eV $= 1.602 \times 10^{-19}$ J
A Joule has units of: [J = kg m^2/s^2]
$h = 6.63 \times 10^{-34}$ J s $= 4.14 \times 10^{-15}$ eV s
$c = 3 \times 10^8$ m/s
$hc = 1240$ eV nm
$m_e c^2 = 0.511$ MeV

For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of a function $f(x)$ is:

$$\langle f(x) \rangle = \int_a^b f(x)\, |\psi(x)|^2 \, dx.$$

You can also make use of following definite integral:

$$\int_0^1 x^m (1-x)^n \, dx = \frac{n!\, m!}{(n+m+1)!}$$