PHOT 222: Quantum Photonics

Final exam: questions (version B)

Michaël Barbier, Spring semester (2024-2025)

General information on the exam

Grading: The final exam counts for 60% of your total grade.

Exam type: The exam consists of 6 open questions/problems. The exam is a written exam and all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the exam is 3 hours.

Exam questions

Please fill in all questions listed below. Each of the questions is valued equally in the score calculation of the exam.

Please tell if any question is unclear or ambiguous.

Question 1: Bohr model

A photon with wavelength 486 nm is emitted from a hydrogen atom. The hydrogen atom thereby transitions from its initial state with principle quantum number n_i to the final state with $n_f = 2$.

(a) What is the energy of the photon?
(b) What is the principal quantum number n_i of the initial state?
(c) Calculate the radius of the electron orbit of initial state n_i (according to Bohr's model).

Question 2: particle in a box in 3D

Consider an electron in an infinite quantum well in 3D (particle in a box) with dimensions $L = L_x = L_y = L_z = 1$ nm.

(a) Calculate the energy level $E_{1,2,3}$ corresponding to the $\psi_{1,2,3}$ state with quantum numbers $n_x = 1$, $n_y = 2$, $n_z = 3$.
(b) Assume the system transitions from the $\psi_{1,2,3}$ state to the ground state $\psi_{1,1,1}$, thereby emitting a photon. What is the frequency of the photon?

Question 3: Quantum hydrogen model

The wave function of a hydrogen atom in the $1s$ state is given by:

$$\psi_{1s}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

(a) Calculate the probability to find the electron within one Bohr radius a_0 from the nucleus (i.e. $r \in [0, a_0]$). *Hints:* The probability for the electron to be in an infinitesimal thin shell volume with radius r is $|\psi(r)|^2 dV = 4\pi|\psi(r)|^2 r^2 dr$. Use integration by parts, see values & formulas.

Question 4: Spin and magnetic moments

A highly excited hydrogen atom (Rydberg atom) is in the $100s$ state ($n = 100$, $l = 0$), and is put in a magnetic field $\vec{B} = B\vec{e}_z$ of 5 Tesla. Consider the $100s$ energy level split by the spin magnetic moment: assume the only effect is the extra potential energy $U = m_s g \mu_B B \approx m_s 2\mu_B B$ where μ_B is the Bohr magneton and you can approximate the gyromagnetic ratio g by 2.

(a) Calculate the energy difference between the $100s$ energy levels for a spin-up and a spin-down state.
(b) The hydrogen atom (with its electron in the $100s$ state) gets ionized by absorbing a photon. What is the maximal wavelength for the photon if the hydrogen atom was in the spin-up state?

Question 5: Spectra of multi-electron atoms

A sodium atom (Na) in the ground state has configuration $1s^2\, 2s^2\, 2p^6\, 3s^1$. The spectrum in the figure shows some of the possible transitions with corresponding (excitation/emission) wavelengths in nm.

(a) What is the energy difference between the $4s$ state of the hydrogen atom and the $4s$ state of the Sodium atom. (calculate using given transition wavelengths and value of $E_{3s} = -5.12$ eV)

(b) Calculate the effective atomic number Z_{eff} (also called effective nuclear charge) for the $4s$ orbital.

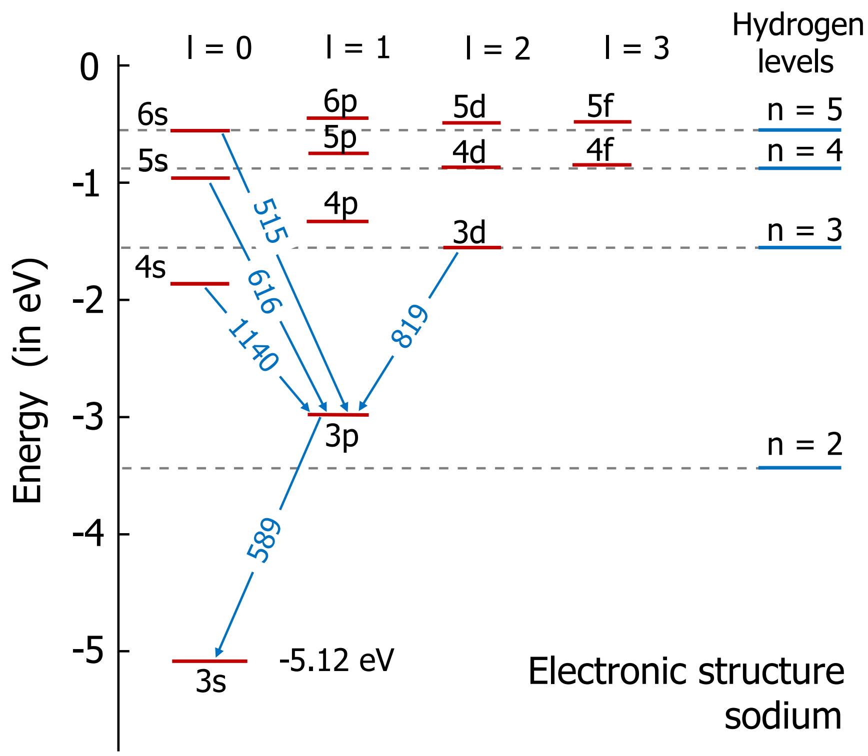

Figure 1: Energy spectrum of sodium for energy-levels from 3s (valence electron in the ground state) and above. The transition wavelengths for emission are shown for some of the allowed transitions (in nm).

Question 6: Molecular spectra

Consider the vibrational states (linear vibration) of the diatomic molecule N_2 existing of two nitrogen atoms. N_2 has a force constant $k' = 2287$ N/m.

(a) Calculate the angular frequencies $\omega = \sqrt{k'/\mu}$ of N_2. For the calculation of the reduced mass μ: remember that nitrogen has an atomic mass of approximately 14 atomic mass units (or Dalton, see formulas/values).
(b) Suppose the molecule is in a state with vibrational quantum number $n = 0$, and transitions to $n = 1$ thereby absorbing a photon. Calculate the wavelength of the photon.

Values and formulas:

Mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of a proton: $m_p \approx 1836\ m_e$
1 eV $= 1.602 \times 10^{-19}$ J
An atomic mass unit: $1\,\text{Dalton} = 1.66 \times 10^{-27}\,\text{kg} \approx \frac{5}{3} \times 10^{-27}\,\text{kg}$
Bohr magneton: $\mu_B = 9.3 \times 10^{-24}$ J/T $= 5.8 \times 10^{-5}$ eV/T
Joule in SI units: $[\text{J} = \text{kg m}^2/\text{s}^2]$
$h = 6.63 \times 10^{-34}$ J s $= 4.14 \times 10^{-15}$ eV s
$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}$ J s $= 6.582 \times 10^{-16}$ eV s
$c = 3 \times 10^8$ m/s
$hc = 1240$ eV nm
Rydberg energy unit: Ry $= 13.6\,$eV
Rydberg constant for hydrogen: $R_H = 1.0968 \times 10^7 \text{m}^{-1} \approx 1.1 \times 10^7 \text{m}^{-1}$
Rydberg constant for heavy atoms: $R_\infty = 1.0974 \times 10^7 \text{m}^{-1} \approx 1.1 \times 10^7 \text{m}^{-1}$
$m_e c^2 = 0.511$ MeV

For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of a function $f(x)$ is:

$$\langle f(x) \rangle = \int_a^b f(x)\, |\psi(x)|^2\, dx.$$

You can also make use of following (in)definite integrals:

Anti-derivatives (indefinite integrals):

$$\int x^n e^{ax}\, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx \qquad \text{for } a \neq 0$$

Definite integrals:

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$
$$\int_0^\infty e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$
$$\int_0^\infty x^2 e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$
$$\int_0^\infty x^4 e^{-ax^2}\, dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$