

PHOT 222: Quantum Photonics

Final exam B: questions & solutions

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Exam Questions

Grading: The final exam counts for 60% of your total grade.

Exam type: Closed-book, all questions can be answered using only pen and paper. Calculators, mobile phones, laptops are not needed, and are not allowed to be used during the exam.

The duration of the exam is 3 hours.

This document contains both the problems and their solutions. Considering the score calculation:

- Each of the questions is valued equally in the total score.
- Subproblems (a , b , c , ...) within a question are weighed equally.
- Answers should contain: The final formula/expression together with its derivation and a numerical approximate value with the **correct units**.

Question 1: Photoelectric effect

When we shine monochromatic light onto a metallic plate, photoelectrons can be emitted from the surface if the frequency of the light is high enough. The table below shows some metals with their corresponding (approximate) workfunction ϕ .

Metal	ϕ (in eV)
Cesium	2.0
Barium	2.5
Lithium	3.0
Silver	4.0
Nickel	5.0

- (a) Suppose the target metal plate is cesium, and blue light with a wavelength of 400 nm, what is the maximum kinetic energy K_{max} that the emitted photoelectrons can acquire?
- (b) Consider the table of metals with their workfunctions ϕ . If we shine blue light

with a wavelength of 400 nm, which of the above metals in the table would exhibit photoemission?

Solution (Q1)

(a) For cesium the workfunction $\phi = 2$ eV, and thus

$$K_{max} = \frac{hc}{\lambda} - \phi = \frac{1240 \text{ eV nm}}{400 \text{ nm}} - 2 \text{ eV} = 1.1 \text{ eV}$$

(b) The metal which have a workfunction $\phi < \frac{hc}{\lambda} = 3.1$ eV (for $\lambda = 400$ nm), as calculated above in (a), would emit photoelectrons. These are the metals: cesium, barium, and lithium.

Question 2: Quantum Harmonic Oscillator

Consider a quantum harmonic oscillator. The eigenenergies E_n for the eigenstates ψ_n are given by $E_n = (n + \frac{1}{2}) \hbar\omega$ with $n = 0, 1, 2, 3, \dots$

(a) If the ground state has energy $E_0 = 1$ eV, what is ω ?

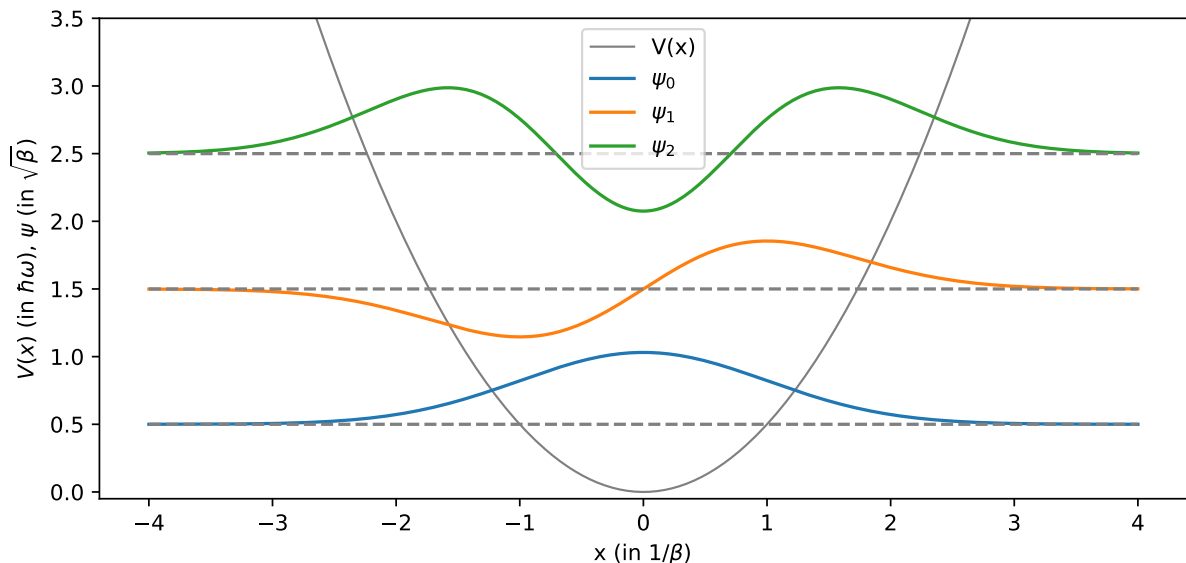
(b) Sketch the potential $V(x)$ and the wave functions $\psi_0(x)$, $\psi_1(x)$, and $\psi_2(x)$.

Solution (Q2)

(a) We extract ω from the definition of the energy levels:

$$E_0 = \frac{1}{2} \hbar\omega \Rightarrow \omega = \frac{2E_0}{\hbar} = \frac{2 \text{ eV}}{6.582 \times 10^{-16} \text{ eV s}} \approx 0.3 \text{ s}^{-1}$$

(b) Below is a rough sketch of the wave functions ψ_0 , ψ_1 , and ψ_2 , shifted to their corresponding energy values E_0 , E_1 , and E_2 .



Question 3: Infinite square well

Assume an electron is in an infinite well with width L . The ground state is given by $\psi_1(x) = A \sin\left(\frac{\pi x}{L}\right)$

- (a) Derive an expression for the normalization constant A .
- (b) What is the expectation value $\langle x^2 \rangle$ for the ground state?

Solution (Q3)

(a) The normalization constant can be found by putting the total probability equal to one:

$$\begin{aligned} 1 &= \int_0^L |\psi_1(x)|^2 dx = |A|^2 \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \\ &= |A|^2 L \int_0^1 \sin^2(\pi u) du, \quad du = \frac{1}{L} dx \\ &= |A|^2 L \frac{1}{2} \end{aligned}$$

Where we used that $\int_0^1 \sin^2(\pi x) dx = 1/2$ (whether we call the variable x or u doesn't matter). Therefore the normalization constant, if chosen real-valued and positive, becomes $A = \sqrt{\frac{2}{L}}$.

(b) For the expectation value $\langle x^2 \rangle$:

$$\begin{aligned} \langle x^2 \rangle &= \int_0^L x^2 |\psi_1(x)|^2 dx = \frac{2}{L} \int_0^1 \sin^2(\pi u) du \\ &= \frac{2}{L} L^3 \int_0^1 u^2 \sin^2(\pi u) du, \quad du = \frac{1}{L} dx \\ &= 2L^2 \left[\frac{1}{6} - \frac{1}{4\pi^2} \right] = L^2 \left[\frac{1}{3} - \frac{1}{2\pi^2} \right] = L^2 \cdot \frac{2\pi^2 - 3}{6\pi^2}, \end{aligned}$$

where we used the formula of the definite integral: $\int_0^1 x^2 \sin^2(\pi x) dx = \frac{1}{6} - \frac{1}{4\pi^2}$ (again: whether we call the variable x or u doesn't matter).

Question 4: Bohr model

A hydrogen atom in the ground state $n = 1$ absorbs a photon of wavelength λ_1 and goes to energy level $n = 3$. It afterwards emits a photon with wavelength λ_2 and goes to level $n = 2$.

- (a) What are the wavelengths of the photon λ_1 and λ_2 ?
- (b) Calculate the velocity of the electron and the orbit radius in the second excited state, that is $n = 3$.

Solution (Q4)

(a) The wavelengths λ_1 and λ_2 can be found by using the formula energy transitions:

$$\frac{1}{\lambda} = \left| \frac{1}{n_i^2} - \frac{1}{n_f^2} \right| R_H \approx \left| \frac{1}{n_i^2} - \frac{1}{n_f^2} \right| 1.1 \cdot 10^{-2} \text{ nm}^{-1}$$

We obtain for λ_1 with $n_i = 1$ and $n_f = 3$:

$$\frac{1}{\lambda_1} = \left| 1 - \frac{1}{9} \right| R_H = \frac{8}{9} R_H \Rightarrow \lambda_1 = \frac{10}{11} \frac{9}{8} 10^2 \text{ nm} = \frac{90}{88} 10^2 \text{ nm} \approx 102 \text{ nm}$$

In a similar fashion we obtain λ_2 :

$$\frac{1}{\lambda_2} = \left| \frac{1}{9} - \frac{1}{4} \right| R_H = \frac{5}{36} R_H \Rightarrow \lambda_2 = \frac{10}{11} \frac{36}{5} 10^2 \text{ nm} = \frac{72}{11} 10^2 \text{ nm} \approx 655 \text{ nm}$$

(b) The orbit radius of the electron is given by $r_n = n^2 a_0$ with $a_0 = 0.0529 \text{ nm}$. For $n = 3$ this becomes: $r_3 = 9a_0 = 0.4761 \text{ nm}$, roughly half a nanometer. For the velocity we can use e.g. the formula for the quantization of the angular momentum $m_e v_n r_n = n\hbar$:

$$v_n = \frac{n\hbar}{m_e r_n} = \frac{n\hbar}{m_e (n^2 a_0)} = \frac{(hc) c}{2\pi m_e c^2 (n a_0)} = \frac{(1240 \text{ eV nm}) c}{2\pi (0.511 \text{ MeV}) (n a_0)} = \frac{1}{n} \frac{5 c}{4\pi 10^3 0.0529}$$

We can approximate this value in units of the speed of light c :

$$v_3 = \frac{5 c}{12\pi 10^3 0.0529} \approx \frac{1}{120\pi} c \approx \frac{1}{360} c$$

where the last value is a very rough approximation (*it corresponds to $2.77 \times 10^{-3} c$ and the real value is $2.43 \times 10^{-3} c$*).

Question 5: Normal Zeeman effect

Assume a hydrogen atom is placed in a magnetic field with $\vec{B}$ along the z-axis. This results in an extra term in the energy $E_n \rightarrow E_n + m_l \mu_B B$ where the Bohr magneton $\mu_B \approx 5.8 \times 10^{-5} \text{ eV/Tesla}$, and m_l is the magnetic orbital quantum number (to simplify the case here we ignore spin).

(a) What is the wavelength corresponding to the $2p$ to $1s$ transition if $B = 0 \text{ Tesla}$?

(b) If we now turn on the magnetic field: in how many lines will the spectral line split? If the splitting between two adjacent lines is about $1.16 \times 10^{-6} \text{ eV}$, what is the required magnitude B of the magnetic field?

Solution (Q5)

(a) For $B = 0$ we have the regular hydrogen atom energies $E_n = -\frac{13.6}{n^2}$ eV, and the transition formula is valid:

$$\frac{1}{\lambda} = \left| \frac{1}{n_i^2} - \frac{1}{n_f^2} \right| R_H \approx \left| \frac{1}{n_i^2} - \frac{1}{n_f^2} \right| 1.1 \times 10^{-2} \text{ nm}^{-1}$$

Thus we obtain for $n_i = 2 \rightarrow n_f = 1$:

$$\frac{1}{\lambda} = \left| \frac{1}{4} - 1 \right| 1.1 \times 10^{-2} \text{ nm}^{-1} = \frac{3}{4} 1.1 \times 10^{-2} \text{ nm}^{-1} \Rightarrow \lambda = \frac{40}{33} \times 10^2 \text{ nm} \approx 121 \text{ nm}$$

(b) Under the assumption that we ignore spin, and only take into account the angular momentum, the line would split in three spectral lines. This can be seen from the fact that the $2p$ orbital has the azimuthal quantum number $l = 1$, and there are three options for m , i.e. $m_l = 0, \pm 1$, resulting in three energy levels: $E_2 + m_l \mu_B B$. Meanwhile the $1s$ orbital has $l = 0$ and $m_l = 0$, meaning that to $E_1 + m_l \mu_B B = E_1$.

For a splitting of 1.16×10^{-6} eV we require $\mu_B B = 1.16 \times 10^{-6}$ eV. Extracting the magnetic field magnitude B we find:

$$B = \frac{1.16 \times 10^{-6} \text{ eV}}{\mu_B} = \frac{1.16 \times 10^{-6} \text{ eV}}{5.8 \times 10^{-5} \text{ eV/T}} = 0.02 \text{ T}$$

where T stands for Tesla.

Question 6: Rotational energy-levels in a diatomic molecule

Assume you have a CO molecule existing of a carbon and an oxygen atom. The rotational energy levels E_J with rotational quantum number $J = 0, 1, 2, 3, \dots$ are given by

$$E_J = \frac{\hbar^2}{2I} J(J+1),$$

with $I = \mu r^2$ the moment of inertia, where r is the inter-atomic distance and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ the reduced mass.

(a) Suppose that a CO molecule with rotational energy E_2 absorbs a photon and transitions to E_3 . If the photon's frequency $f \approx 3.5 \times 10^{11}$ Hz, what is then the moment of inertia I of the CO molecule?

(b) Derive the inter-atomic distance r (distance between the C and the O atom) from the value of I . Remember that carbon has an atomic mass of approximately 12 and oxygen 16 atomic mass units (or Dalton, see formulas/values).

Solution (Q6)

(a) The transition energy should equal the photon energy: $hf = E_3 - E_2$ and is given by

$$hf = E_3 - E_2 = \frac{\hbar^2}{2I} (3(3+1) - 2(2+1)) = \frac{3\hbar^2}{I}$$

We can then extract the moment of inertia I :

$$I = \frac{3\hbar^2}{hf} = \frac{3\hbar}{2\pi f} = \frac{3\hbar}{2\pi (3.5 \times 10^{11}) \text{ Hz}} = \frac{3 (1.055 \times 10^{-34} \text{ J s})}{2\pi (3.5 \times 10^{11} \text{ s}^{-1})} \approx \frac{1}{7} \times 10^{-45} \text{ kg m}^2$$

where we used that $\text{J} = \text{kg m}^2 \text{s}^{-2}$ and $\text{Hz} = \text{s}^{-1}$.

(b) The interatomic distance r can be extracted from $I = \mu r^2$. First we calculate μ :

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{(12 \text{ Dalton}) \cdot (16 \text{ Dalton})}{12 \text{ Dalton} + 16 \text{ Dalton}} = \frac{192}{28} \text{ Dalton} = \frac{48}{7} \text{ Dalton}$$

$$r = \sqrt{\frac{I}{\mu}} \approx \sqrt{\frac{\frac{1}{7} \times 10^{-45} \text{ kg m}^2}{\frac{48}{7} \text{ Dalton}}} \approx \sqrt{\frac{\frac{1}{7} \times 10^{-45} \text{ kg m}^2}{\frac{48}{7} \cdot \frac{5}{3} \times 10^{-27} \text{ kg}}} \approx \sqrt{\frac{1}{90}} \times 10^{-9} \text{ m} \approx 0.1 \text{ nm}$$

where the last is a rough approximation but gives us the expected order of about one angstrom (the real value is 0.1128 nm).

Values and formulas:

Mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg

Mass of a proton: $m_p \approx 1836 m_e$

1 eV = 1.602×10^{-19} J

An atomic mass unit: 1 Dalton = 1.66×10^{-27} kg $\approx \frac{5}{3} \times 10^{-27}$ kg

Bohr magneton: $\mu_B = 9.3 \times 10^{-24}$ J/T = 5.8×10^{-5} eV/T

Joule in SI units: [J = kg m²/s²]

$h = 6.63 \times 10^{-34}$ J s = 4.14×10^{-15} eV s

$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}$ J s = 6.582×10^{-16} eV s

$c = 3 \times 10^8$ m/s

$hc = 1240$ eV nm

Rydberg energy unit: Ry = 13.6 eV

Rydberg constant for hydrogen: $R_H = 1.0968 \times 10^7 \text{m}^{-1} \approx 1.1 \times 10^7 \text{m}^{-1}$

Rydberg constant for heavy atoms: $R_\infty = 1.0974 \times 10^7 \text{m}^{-1} \approx 1.1 \times 10^7 \text{m}^{-1}$

$m_e c^2 = 0.511$ MeV

For a wave function $\psi(x)$ with $x \in [a, b]$, the expectation value of a function $f(x)$ is:

$$\langle f(x) \rangle = \int_a^b f(x) |\psi(x)|^2 dx.$$

Definite integrals:

$$\begin{aligned}\int_0^1 \sin^2(\pi x) dx &= \frac{1}{2} \\ \int_0^1 x \sin^2(\pi x) dx &= \frac{1}{4} \\ \int_0^1 x^2 \sin^2(\pi x) dx &= \frac{1}{6} - \frac{1}{4\pi^2} \\ \int_0^1 x^3 \sin^2(\pi x) dx &= \frac{1}{8} - \frac{3}{8\pi^2}\end{aligned}$$