Question (100 points)

Consider the following composite network element.

where the blue interaction is the inhibition of \( X \) by \( Y \) binding to form the complex \( YX \) via

\[
Y + X \leftrightarrow YX
\]

with a forward rate constant of \( k \) and a backward rate constant of \( k' \).

a) (50 points) Supposing that the gene transcription mechanism operates with an inherent time delay of \( T = 3 \) units of time, determine the dynamic behavior of this system by numerically solving the corresponding system of ordinary differential equations using the following system parameters

\[
\begin{align*}
\beta_X &= \alpha_X = \beta_Y = \alpha_Y = 1 \\
\kappa_{XX} &= \kappa_{XY} = 0.4 \\
n_X &= n_Y = 3 \\
k &= 100 \\
k' &= 1
\end{align*}
\]

for \( t \in [0,50] \) with the initial values of \( ([X])'(0^-) = 1, ([Y])'(0^-) = 0 \) and \( ([YX])'(0^-) = 0 \).

(Hint: The inherent time delay in the gene transcription mechanism links the current rate of change in protein concentrations to the transcription factor concentrations \( T \) units of time in the past.)

b) (25 points) By repeating the numerical simulation for varying \( T \), draw the graph of the period of the system at the limit cycle with respect to time delay.

(Hint: Make sure that the time delays are large enough to produce a stable oscillatory behavior.)

c) (25 points) Plot the transient behavior of the state variables in pairs as well as all together in the full state space for one period of the oscillation cycle.
(Hint: You may want to consider a nonlinear logarithmic-like scaling of the axes that would provide a better resolution in the lower concentrations.)