J. Differential Equations 251 (2011) 1841–1863

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Uniform decay rates for the energy of weakly damped defocusing semilinear Schrödinger equations with inhomogeneous Dirichlet boundary control

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ARTICLE INFO

Article history:
Received 25 November 2010
Revised 4 April 2011
Available online 16 April 2011

Keywords:
Nonlinear Schrödinger equation
Inhomogeneous Dirichlet boundary condition
Existence
Stabilization
Boundary control
Direct multiplier method
Monotone operator theory
Compactness

ABSTRACT

In this paper, we study the open loop stabilization as well as the existence and regularity of solutions of the weakly damped defocusing semilinear Schrödinger equation with an inhomogeneous Dirichlet boundary control. First of all, we prove the global existence of weak solutions at the H^1-energy level together with the stabilization in the same sense. It is then deduced that the decay rate of the boundary data controls the decay rate of the solutions up to an exponential rate. Secondly, we prove some regularity and stabilization results for the strong solutions in H^2-sense. The proof uses the direct multiplier method combined with monotonicity and compactness techniques. The result for weak solutions is strong in the sense that it is independent of the dimension of the domain, the power of the nonlinearity, and the smallness of the initial data. However, the regularity and stabilization of strong solutions are obtained only in low dimensions with small initial and boundary data.

1. Introduction

Nonlinear Schrödinger equations (NLS) have been extensively studied in the last few decades with motivations coming from numerous physical applications that include nonlinear models in plasma

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.04.003

physics and fiber optics. The studies on NLS are naturally divided into two categories: the Cauchy problem in $\mathbb{R}^n$ and the initial–boundary value problems defined on a general domain $\Omega \subset \mathbb{R}^n$. This split is rather natural due to vast differences in qualitative properties of Schrödinger operator defined on R^n rather than on bounded domain. Indeed, it is well known that solutions of the Schrödinger equation in R^n display some regularizing effects, while the same effect is unknown when considering bounded domains. Not surprisingly, the techniques developed for studying the problem are also very different. In the case of the Cauchy problem, Strichartz type estimates are helpful for obtaining many results. However, these types of estimates do not have the same power and applicability when considering solutions of the initial–boundary value problems. Therefore fewer results are available in the latter case. Moreover, most of the results which pertain to general domains assume very standard types of conditions such as homogeneous boundary data, strong assumptions on the dimension of the domain, the power of the nonlinearity, or the size of the initial data.

It is the aim of this paper to consider NLS in the context of bounded domain in R^n with *non-homogeneous boundary inputs of Dirichlet type*. Very few results are available for this type of dynamics, with majority of developments carried out in one-dimensional settings.

Our primary focus in this paper is existence, regularity and long time behavior of the corresponding dynamics. We shall derive decay rates for the energy of the solutions that are uniform with respect to the energy of the initial data and of boundary inputs. In the language of control theory one may state that open loop uniform stabilization is achieved by using classical weak dissipation acting on the equation with suitably decaying boundary data.

We will consider the following weakly damped semilinear Schrödinger equation,

$$iu_t - \Delta u + f\left(|u|^2\right)u + iau = 0 \text{ in } \Omega_\infty := \Omega \times (0, \infty), \tag{1}$$

with initial data and inhomogeneous Dirichlet boundary condition

$$\begin{cases} u = Q & \text{on } \Gamma_\infty := \Gamma \times (0, \infty), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary Γ, a is a nonnegative constant, $f(s) = gs^{\frac{p}{2}}$, where $g > 0$ and $p > 0$ are constants, $Q : \Gamma_\infty \to \mathbb{C}$, $u_0 : \Omega \to \mathbb{C}$. The case $a = 0$ is simply the classical nonlinear Schrödinger equation.

The decay rates for weakly damped semilinear focusing and defocusing Schrödinger equations with *zero Dirichlet boundary condition* (i.e., $Q \equiv 0$) have been considered by Tsutsumi [8] where exponential stabilization of H^1-solutions for $0 < p < \infty$ in dimensions $n = 1, 2$, and for $0 < p < 4/(n-2)$ in dimensions $n \geqslant 3$ is proved. Smallness of initial data for either cases if $p \geqslant 4/n$ is also assumed in [8]. In addition [8] proves stabilization in H^{2k}-topology for H^{2k}-solutions where k is an integer (greater than $n/4$) and the initial data are suitably small.

One may quickly notice that in the defocusing case ($g > 0$) with zero boundary data ($Q \equiv 0$), the (exponential) stabilization follows quickly from mass and energy estimates,

$$\frac{d}{dt}\|u\|^2 = -2a\|u\|^2 \quad \left(\|\cdot\| := \|\cdot\|_{L^2(\Omega)}\right)$$

and

$$\frac{d}{dt}E = -aE - \frac{agp}{p+2}\|u\|^{p+2}_{L^{p+2}(\Omega)} \leqslant -aE,$$

where the energy is defined as

$$E := \frac{1}{2}\|\nabla u\|^2 + \frac{g}{p+2}\|u\|^{p+2}_{L^{p+2}(\Omega)}.$$

Therefore, the assumptions in [8] that relate the power of the nonlinearity p to the dimension n and smallness of initial data u_0 are actually essential only for the stabilization of the focusing case ($g < 0$) at the H^1-level.

However, in the case of inhomogeneous Dirichlet boundary condition, the equation in the defocusing structure is far from yielding trivial energy identities. These identities *include boundary integrals that involve the directional derivative* of the solution. The latter are not defined on the energy space considered. One can see this via formal calculation

$$\frac{d}{dt}\frac{1}{2}\|u\|^2 = -a\|u\|^2 + \operatorname{Im}\langle \nabla u \cdot \nu, Q\rangle \quad \left(\langle u, v\rangle := \int_\Gamma u\bar{v}\, d\Gamma\right)$$

and

$$\frac{d}{dt}E = -aE - \frac{agp}{p+2}\|u\|^{p+2}_{L^{p+2}(\Omega)} + \operatorname{Re}\langle \nabla u \cdot \nu, aQ + Q_t\rangle$$

in the case of an inhomogeneous data on the boundary.

The insight we obtain from the homogeneous counterpart tells us that the stabilization statement could still be valid depending on the behavior of the given boundary control. Actually, this is what we will prove in this paper, and we will show that we are able to generalize the result of stabilization in the defocusing case by considering inhomogeneous Dirichlet data on the boundary. Our result will not impose any assumptions on the power of the nonlinearity, the dimension of the space, or the smallness of the initial data at H^1-level. However, higher regularity analysis will require restrictions involving the dimension of the space and the size of the initial and boundary data.

Few words about the literature. Existence of solutions for (1)–(2) is not a trivial problem due to the simultaneous presence of low regularity boundary data and nonlinearity. Standard methods of homogenization do not apply and the analysis involved is more subtle. In fact, the same problem without the weak damping term *iau* has been considered in the paper of Strauss and Bu [14]. The result obtained in [14] states that there is at least one weak solution which belongs to the class of functions $L^\infty_{\mathrm{loc}}(0,\infty; H^1(\Omega)\cap L^{p+2}(\Omega))$ for given initial data $u_0 \in H^1(\Omega)$ (actually $H^1(\Omega)\cap L^{p+2}(\Omega)$) and a boundary data $Q \in C^3([0,\infty)\times\Gamma)$ which has compact support in Γ. Later Bu, Tsutaya, and Zhang in [1] proved that the same result also holds for the focusing problem where $p < 2/n$. The approach carried out in both papers relies on constructing explicit Lipschitz approximations of the nonlinear term $|u|^p u$ and then solving the truncated problems via fixed point theorem. The corresponding truncated solutions are shown to converge to the sought-after solutions by exploiting rather involved compactness argument. The results obtained are of local character, without taking into consideration long time behavior.

Instead, our interest is also in asymptotic time behavior. The methods we used are very different from the methods used in the mentioned papers. Since our focus is on long time behavior, the approach taken is based on forcing monotone behavior in the suitably selected approximations. By doing so we will be able to obtain simultaneous existence and stabilization. Our proof will proceed through two main steps. In the first step, we homogenize the nonlinear problem by extending the boundary data into the domain by using recent sharp results from the linear theory. In the second step, we will resort to the theory of monotone operators together with the direct multiplier method, perturbed energy and compactness methods. Our approach has several advantages: (i) it provides sharper than in [14] results in terms of the regularity imposed on the boundary data, (ii) it applies to the analysis of long time behavior, including the analysis of decay rates, (iii) the methodology is intrinsic and can also be applied to more general types of nonlinearities where it is difficult to construct explicit Lipschitz approximations and run the machinery of fixed points.

NLS with Dirichlet condition is also important from the point of boundary control problems. The study of the corresponding exact boundary controllability problem has been recently initiated in the papers of Rosier and Zhang, [12,13], Deng and Yiao, [2] and in Zong, Zhao, Yin, and Chi [15]. However, these results are special in the sense that they are either one-dimensional, local or they apply to very special geometries. However, the general exact boundary controllability problem for NLS is still open.

2. Main results

We shall employ the following variational definition of *weak solution*:

Definition. Given initial and boundary data, $u_0 \in H^1(\Omega)$, $Q \in H^{1,1}(\Gamma_\infty)$, u is called a *weak solution* of the problem (1)–(2) if $u \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega) \cap L^{p+2}(\Omega))$, $\gamma_{t=0} u = u_0$ in $L^2(\Omega)$, $\gamma_{\partial\Omega} u = Q$ in $L^2(0,T; H^{1/2}(\Gamma))$ and

$$\int_0^T -(u, \varphi_t) + i(\nabla u, \nabla \varphi) + i\big[|u|^p u, \varphi\big]_{L^{(p+2)'}, L^{(p+2)}} - a(u, \varphi)\, dt = 0$$

holds true for each $T > 0$ and for all $\varphi \in C_0^\infty((0,T); H_0^1(\Omega) \cap L^{p+2}(\Omega))$.

Notation. $(u, v) := \int_\Omega u \bar{v}\, dx$. $\gamma_{t=0}$ is the trace operator to $\Omega \times \{0\}$ and $\gamma_{\partial\Omega}$ is the trace operator to $\partial\Omega$ respectively. $[\cdot,\cdot]_{X',X}$ is the Banach space pairing.

Based on the above notion of the solution, we have the following existence and long time behavior results for the solutions controlled from the boundary.

Theorem 1. *Let $u_0 \in H^1(\Omega) \cap L^{p+2}(\Omega)$, $Q \in H^{1,1}(\Gamma_\infty) \cap L^{p+2}(0,\infty; L^{p+2}(\Gamma))$ and the compatibility condition $u_0|_\Gamma \equiv Q(0)$ be satisfied. Then there exists a weak solution u of the problem* (1)–(2)*. Moreover, for any $b < a$, there exists a positive constant $C = C(b) > 0$ such that*

$$\big\|u(t)\big\|_{H^1(\Omega)\cap L^{p+2}(\Omega)} \leqslant C e^{-bt} \Upsilon(t)$$

where

$$\Upsilon(t) := \left(\|u_0\|^2_{H^1(\Omega)\cap L^{p+2}(\Omega)} + \int_0^t e^{2bs} \big\| Q(s) \big\|_w^2\, ds \right)^{\frac{1}{2}}$$

and

$$\|Q\|_w := \left(\|Q\|^2_{H^1(\Gamma)} + \|Q_t\|^2_{L^2(\Gamma)} + \|Q\|^{p+2}_{L^{p+2}(\Gamma)} \right)^{\frac{1}{2}}.$$

Remark 1. For the global existence of weak solutions, it is actually enough that $Q \in H^{1,1}(\Gamma_T) \cap L^{p+2}(0,T; L^{p+2}(\Gamma))$ for each $T > 0$. However, if we want solutions to decay to zero we in addition need $\|Q(t)\|_w \to 0$ as $t \to \infty$.

Remark 2. We notice that in the estimate of Theorem 1, the boundary data contributes to the decay up to an exponential rate. Thus, the fastest rate of decay that we get by the estimate in Theorem 1 is given by

$$\max\big\{O\big(e^{-2bt}\big), O\big(\|Q\|_w\big)\big\}.$$

Therefore, in order to obtain the exponential stabilization of the solutions it is enough to control the system with an exponentially decaying boundary data.

Remark 3. Note that, due to the Sobolev trace inequality

$$\|Q(t)\|_{L^2(\Gamma)} \leqslant C\|u(t)\|_{H^1(\Omega)}.$$

Thus if $\|Q(t)\|_{L^2(\Gamma)} \to \infty$ as $t \to T^*$ for some $T^* > 0$, then it follows that the weak solutions will blow up at energy level. It would be interesting to look for weaker assumptions that prevent global existence of the solutions. Such conditions are known for the focusing case.

Remark 4. The result stated in Theorem 1 improves related result in [14]. Indeed, the existence result obtained in [14] requires much higher regularity of the boundary data (C^3 instead of C^1). In addition, the result presented in Theorem 1 provides decay rates for the corresponding solutions. This latter topic is not considered in [14].

Remark 5. Exponential decay of solutions with boundary feedback operator has been studied by several authors. However, this class of problems is technically different, due to the absence of inhomogeneous effects on the boundary.

We will consider next behavior of strong solutions. We shall establish both local and global existence along with decay rates of strong solutions for the problem (1)–(2) in dimensions $n = 1, 2, 3$. The following notion of strong solutions will be used.

Definition. A function u is said to be a *strong solution* of the problem (1)–(2) if

$$u \in C\big([0, T_{\max}); H^2(\Omega)\big) \cap C^1\big([0, T_{\max}); L^2(\Omega)\big)$$

and satisfies the equation

$$iu_t - \Delta u + f\big(|u|^2\big)u + iau = 0$$

in $L^2(\Omega)$ for all $t \in (0, T_{\max})$ and $u(0) = u_0$. If $T_{\max} < \infty$, u will be called *local* strong solution and if $T_{\max} = \infty$, u will be called *global* strong solution.

The following local and global results are given in Theorems 2 and 3, where the global existence result assumes smallness of initial and boundary data in addition to the restriction on the dimension of the domain.

Theorem 2. *Let $n \leqslant 3$. Then, the problem* (1)–(2) *with control function $Q \in H^{2,2}(\Gamma_\infty)$ satisfying the compatibility condition $Q(0) = u_0|_\Gamma$ and having the initial data $u_0 \in H^2(\Omega)$ possesses a unique local strong solution on an interval* $[0, T_{\max})$. *Moreover, if $T_{\max} < \infty$, then*

$$\limsup_{t \to T_{\max}} \|u(t)\|_{L^\infty(\Omega)} = \infty.$$

Theorem 3. *Let $n \leqslant 3$, $u_0 \in H^2(\Omega)$ and $Q \in H^{2,2}(\Gamma_\infty)$ together with the compatibility condition $u_0|_\Gamma = Q(0)$ and suppose Q and u_0 are small enough* (*in the sense of inequalities* (44)–(45)). *Then, there exists a unique global strong solution u to the problem* (1)–(2). *Moreover, for any $b < a$, there is some $C = C(b) > 0$ such that*

$$\|u(t)\|_{H^2(\Omega)} \leqslant Ce^{-bt}.$$

Remark 6. For $n = 1$, the smallness assumption on the initial data needed for global existence of solutions is redundant since we already know the local solution is global in H^1-sense via Theorem 1. Indeed, we have the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and we use the blow-up alternative proved in Theorem 2 to show that the maximal time of existence is infinite.

3. Linear theory and orientation

Our proof relies on two fundamental ingredients: (i) dynamic extension of boundary data into the interior while preserving the optimal regularity, (ii) use of maximal monotone operator theory in constructing suitable approximations of solutions to nonlinear problem. The limits of these approximations will be shown to satisfy weak form of the equation.

The first ingredient is due to relatively recent developments in linear boundary control theory developed for Schrödinger equations. Relevant results will be summarized below. The second ingredient is motivated by recent study of monotone operator theory in the context of Ginzburg–Landau equations, see for example Okazawa and Yokota [10] and subsequent papers by the same authors.

3.1. Linear theory

Regarding the existence of solutions for the *linear* Schrödinger equation, the following regularity result is known and will be used [5, Theorem 10.9.7.1].

Theorem 4. *Given an inhomogeneous Dirichlet boundary control $Q \in H^{1,1}(\Gamma_T)$ and initial data $u_0 \in H^1(\Omega)$ together with the compatibility condition $u_0|_\Gamma = Q(0)$, there exists a unique solution v which belongs to the class of functions $C([0,T]; H^1(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ for the problem*

$$\begin{cases} i\dfrac{dv}{dt} - \Delta v = 0 & \text{in } \Omega_T, \\ v = Q & \text{on } \Gamma_T, \\ v(0) = u_0 & \text{in } \Omega. \end{cases} \tag{3}$$

Moreover, the solution continuously depends on the given data. Namely, the mapping

$$u_0, Q \to v, v_t, \frac{\partial v}{\partial \nu}$$

is continuous from the space

$$H^1(\Omega) \times H^{1,1}(\Gamma_T)$$

into the space

$$C\big([0,T]; H^1(\Omega)\big) \times C\big([0,T]; H^{-1}(\Omega)\big) \times L^2(\Gamma_T).$$

Notation. $H^{r,s}(\Gamma_T) := L^2(0,T; H^r(\Gamma)) \cap H^s(0,T; L^2(\Gamma))$, $T \leqslant \infty$.

Remark 7. Note "hidden" regularity of the normal derivative of solutions. These are typically not defined on $H^1(\Omega)$ space. The result of Theorem 4 shows that the normal traces are defined in $L^2(\Gamma_T)$ for H^1-Schrödinger solutions. This fact will be used critically in the development.

Theorem 4 allows to extend the inhomogeneous boundary data Q as the unique solution of the linear Schrödinger equation in (3) with the same initial and boundary data – preserving optimal Schrödinger regularity. This device is applied to homogenize the nonlinear problem. There are two advantages of selecting this particular extension. First of all, this helps us to get a clear presentation for the homogenized nonlinear equation such as zero initial and boundary conditions – without the forcing terms. Secondly, this extension provides smooth solutions that are C^1-functions. This allows for preservation of time regularity of the nonlinearity, which property is not enjoyed when homogenization uses an arbitrary (say – standard elliptic) extension.

In the process of the proof the following result pertaining to non-autonomous problems with Lipschitz nonlinearity will be used.

Lemma 1. *Consider the evolution equation*

$$\begin{cases} \dfrac{dw}{dt} + Aw + L(t, w) = 0, \\ w(0) = w_0 \end{cases} \tag{4}$$

on the complex Hilbert space $H := L^2(\Omega)$. $A := i\Delta : D(A) \subset H \to H$ *with* $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$, $L(t, w) = F(v + w)(t)$ *where* $F : H \to H$ *Lipschitz,* $v \in C([0, T]; H^2(\Omega)) \cap C^1([0, T]; L^2(\Omega))$. *Moreover, let* $w_0 \in D(A)$. *Then, problem* (4) *has a unique solution* w *which belongs to the class of functions* $C^1([0, T]; H) \cap C([0, T]; D(A))$.

Proof. Since $v \in C^1([0, T]; L^2(\Omega))$, we have $\|v_t\| < M_T := \sup_{0 \leqslant t \leqslant T} \|v_t\|$. Hence,

$$\left\| v(t_2) - v(t_1) \right\| = \left\| \int_{t_1}^{t_2} \frac{dv}{dt} \, dt \right\| \leqslant \int_{t_1}^{t_2} \left\| \frac{dv}{dt} \right\| dt \leqslant M_T |t_2 - t_1|,$$

i.e., v is Lipschitz on $[0, T]$. Then, since A is linear m-accretive, by the classical theory of abstract semilinear evolution equations, see for example Pazy [3, Section 6.1], one can obtain that L is Lipschitz in time variable on $[0, T]$ and problem (4) has a unique mild solution which is also classical. □

3.2. Orientation

A common strategy for proving existence of solutions to nonlinear PDEs relies on the following standard steps: (1) construct a suitable approximation of the sought-after solution, (2) establish appropriate a priori bounds, and finally (3) pass with the limit.

In the present situation the construction of a suitable approximation is a subtle point. Reconciling Dirichlet low regularity boundary data with nonlinearity is not obvious. The reason for this is intrinsic lack of compatibility between the multipliers needed for the treatment of Dirichlet boundary conditions and superlinear nonlinearity – as in the case of the present paper. In fact, this issue was already present in [14], where the authors resorted to certain truncation type of approximants effective for more regular boundary data.

In our case, when dealing only with $H^{1,1}$ boundary data, we will rely on monotone type of approximation (though the problem is not monotone) which will allow for estimates effective not only in the context of existence but also in the context of asymptotic behavior.

In what follows below we provide qualitative description for the strategy used in the proof of the main results.

We will be looking for a solution u of the following form $u := v + w$ where v satisfies (3) and w satisfies the following problem *homogeneous* on the boundary and driven by the nonlinear term depending on v:

$$\begin{cases} i\dfrac{dw}{dt} - \Delta w + F_v(w) = 0 & \text{on } \Omega_\infty, \\ w = 0 & \text{on } \Gamma_\infty, \\ w(0) = 0 & \text{on } \Omega, \end{cases} \tag{5}$$

with

$$F_v(w) = f\left(|w+v|^2\right)(w+v) + ia(w+v).$$

Thus, our present task is the following: given u_0, Q solve coupled system of equations in the variables v, w described by (3) and (5). Since solution to v is provided by Theorem 4, the main task is to establish solvability of (5) for a given v – with a specified – by Theorem 4 – regularity. We note here, that Eq. (5) while homogeneous on the boundary, does not have good nonlinear structure amenable to a priori bounds. This precludes direct study of the evolution defined by w. In order to handle this the following strategy will be employed.

- *Step 1*: Consider approximation of w equation, defined for w_n by (5) with $F_v(w)$ replaced by a suitable $F_{n,v_n}(w_n)$. The nonlinear term $F_{n,v_n}(w_n)$ involves Yosida type of approximation and smooth approximant v_n of solution v to (5). This latter approximant is constructed by taking the data $u_0 = u_{0,n}$ and $Q = Q_n$ sufficiently smooth.
- *Step 2*: For each value of the parameter n and selected v_n prove unique existence and regularity of solution w_n. This will be done with the help of Lemma 1, on the strength of Lipschitz property of Yosida approximation and regularity of dynamic extension v_n.
- *Step 3*: Define $u_n \equiv v_n + w_n$. This is the sought-after approximation of the original problem.
- *Step 4*: Prove appropriate a priori estimates for u_n. This is the most technical part of the argument which involves, among other things, showing that conservative effect of superlinear term integrated against flux multipliers is partially invariant under Yosida approximation (i.e., the latter has some cancellation properties). This property is shown in Lemma 5.
- *Step 5*: Establish weak limits of the corresponding subsequences and show that this limit satisfies weak form of equation. Multipliers used in controllability theory of Schrödinger equation are essentially used for this step.
- *Step 6*: Asymptotic behavior of solutions u_n. Here again, controllability multipliers are critical. Finally weak lower-semicontinuity argument establishes the same asymptotic behavior for the sought-after solution u of our original problem given by (1)–(2).

Remark 8. As far as we know, the uniqueness of weak solutions is still an open question. In the case of regular solutions, uniqueness is shown by Theorem 2.

4. Proof of Theorem 1 – weak and exponentially decaying solutions

This section is devoted to the proof of Theorem 1.

Proof. Note that we cannot directly apply the result of Lemma 1 to problem (5) since the nonlinear part is not Lipschitz. However, we can try to construct approximate solutions to problem (5) by replacing the nonlinearity with its Yosida approximations. We are able to use the Yosida approximation scheme here due to the following lemma which says that the nonlinearity in (5) is monotone in the space H. This has been proved in Okazawa and Yokota [9, Lemma 3.1].

Lemma 2. *Let $B : D(B) \subset H \to H$ be the operator defined by $Bu := |u|^p u$ with $D(B) := \{u \in H;\ |u|^p u \in H\}$. Then, B is m-accretive.*

Thus, we can define the (Lipschitz continuous) Yosida approximations B_n of B in terms of the resolvents J_n, namely,

$$J_n := \left(1 + \frac{1}{n}B\right)^{-1} \quad \text{and} \quad B_n := n(I - J_n) = BJ_n.$$

Moreover, from the general theory of monotone operators it is known that we can represent the operators B and B_n by subdifferentials of ψ and ψ_n given as

$$\psi(u) := \begin{cases} \frac{1}{p+2}\|u\|^{p+2}_{L^{p+2}(\Omega)} & \text{for } u \in L^{p+2}(\Omega), \\ \infty & \text{otherwise} \end{cases}$$

and

$$\psi_n(u) := \min_{v \in H}\left\{\frac{n}{2}\|v - u\|^2 + \psi(v)\right\} = \frac{1}{2n}\|B_n u\|^2 + \psi(J_n) \quad \text{for } u \in H$$

so that

$$B = \partial\psi \quad \text{and} \quad B_n = \partial\psi_n.$$

Now, consider the following approximate problems:

$$\begin{cases} \dfrac{dw_n}{dt} - iS(w_n) + F_{n,v_n}(w_n) = 0 \quad \text{on } (0, \infty), \\ w_n(0) = 0 \end{cases} \tag{6}$$

where $S = -\Delta$ with zero Dirichlet boundary conditions,

$$F_{n,v_n}(w_n) := -igB_n(w_n + v_n) + a(w_n + v_n),$$

and $v_n \in C^1([0, \infty); L^2(\Omega))$ is an approximation to the unique extension of the boundary data into the region Ω, namely v_n's are the solutions of (3), u_0 is replaced by $H^2(\Omega) \cap L^{p+2}(\Omega)$ functions $u_{n0} \to u_0$ in $H^1(\Omega) \cap L^{p+2}(\Omega)$, and Q is replaced by $H^{2,2}(\Gamma_T)$ functions $Q_n \to Q$ in $H^{1,1}(\Gamma_T)$, so that $C([0, T]; H^2(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \ni v_n \to v$ in $C([0, T]; H^1(\Omega))$ and $\frac{d}{dt}v_n \to \frac{dv}{dt}$ in $C([0, T]; H^{-1}(\Omega))$. In order to construct such v_n, we use Theorems 4 and 5. Then (6) satisfies the conditions of problem (4) and there exists a solution $w_n \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H^2(\Omega) \cap H^1_0(\Omega))$, which solves (6) for each n.

Now, if we define $u_n := w_n + v_n$, then $u_n \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H^2(\Omega))$ solves the following problem,

$$\begin{cases} i\dfrac{du_n}{dt} - \Delta u_n + gB_n(u_n) + iau_n = 0 \quad \text{on } (0, \infty), \\ u_n(0) = u_{n0} \end{cases} \tag{7}$$

together with $u_n|_\Gamma = Q_n$. This is an approximation to our original problem. Now, we will obtain suitable estimates on the approximate solutions u_n, in order to be able to pass to a subsequence which converges to the solution of our original problem. Taking L^2-inner product of (7) with u_n, looking at the imaginary parts, we compute

$$\operatorname{Re}(u_{nt}, u_n) + \overbrace{\operatorname{Im}(\nabla u_n, \nabla u_n)}^{0} - \operatorname{Im}\langle \nabla u_n \cdot \nu, Q_n\rangle + \overbrace{\operatorname{Im}(B_n u_n, u_n)}^{0} + a(u_n, u_n) = 0, \tag{8}$$

where ν is the unit outward vector. Note that $\operatorname{Im}(B_n u_n, u_n) = 0$ because

$$\begin{aligned}(B_n u_n, u_n) &= \left(B_n u_n, \frac{1}{n} B_n u_n + J_n u_n\right) = (B J_n u_n, J_n u_n) + \frac{1}{n}\|B_n u_n\|^2 \\ &= p\psi(J_n u_n) + 2\psi_n(u_n) \geqslant 0.\end{aligned} \tag{9}$$

On the other hand, we have

$$\operatorname{Re}(u_{nt}, u_n) = \frac{1}{2}\frac{d}{dt}\|u_n\|^2.$$

Hence, we can rewrite (8) as

$$\frac{d}{dt}\frac{1}{2}\|u_n\|^2 = -a\|u_n\|^2 + \operatorname{Im}\langle \nabla u_n \cdot \nu, Q_n\rangle. \tag{10}$$

Taking L^2-inner product of (7) with u_{nt}, looking at the real parts, we have

$$\overbrace{\operatorname{Re}(iu_{nt}, u_{nt})}^{0} + \operatorname{Re}(\nabla u_n, \nabla u_{nt}) - \operatorname{Re}\langle \nabla u_n \cdot \nu, Q_{nt}\rangle + g\operatorname{Re}(B_n u_n, u_{nt}) + a\operatorname{Re}(iu_n, u_{nt}) = 0. \tag{11}$$

By using (7), we can rewrite the term $a\operatorname{Re}(iu_n, u_{nt})$ as

$$a\operatorname{Re}(iu_n, u_{nt}) = a(\nabla u_n, \nabla u_n) - a\operatorname{Re}\langle \nabla u_n \cdot \nu, Q_n\rangle + ag\operatorname{Re}(B_n u_n, u_n) + a\overbrace{\operatorname{Re}(iau_n, u_n)}^{0}. \tag{12}$$

On the other hand, we have

$$\operatorname{Re}(\nabla u_n, \nabla u_{nt}) = \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2. \tag{13}$$

Now, we recall the following classical lemma given in the book of Showalter [11, Chapter IV, Lemma 4.3].

Lemma 3. *Let $\varphi : H \to (-\infty, \infty]$ be proper, convex, and lower-semicontinuous on the Hilbert space H with subgradient $\partial\varphi$. If $u, \frac{du}{dt} \in L^2(0, T; H)$ and if there exists a $g \in L^2(0, T; H)$ with $g \in \partial\varphi(u)$ a.e. on $[0, T]$, then the function $\varphi \circ u$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\varphi\big(u(t)\big) = \operatorname{Re}\left(h(t), \frac{du}{dt}(t)\right), \quad \text{a.e. } t \in [0, T]$$

for any function h with $h \in \partial\phi(u)$ a.e. on $[0, T]$.

Since, ψ_n, u_n, and u_{nt} satisfy the conditions of Lemma 3 together with $B_n u_n = \partial\psi_n(u_n)$, we can conclude that

$$\operatorname{Re}(B_n u_n, u_{nt}) = \frac{d}{dt}\psi_n(u_n). \tag{14}$$

Using (12)–(14), we can rewrite (11) as

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u_n\|^2 + g\psi_n(u_n)\right) = -2a\left(\frac{1}{2}\|\nabla u_n\|^2 + g\psi_n(u_n)\right) + \operatorname{Re}\langle\nabla u_n\cdot\nu, aQ_n + Q_{nt}\rangle - \frac{agp}{p+2}\|J_n u_n\|^{p+2}_{L^{p+2}(\Omega)}. \tag{15}$$

Up to this point, we have used the standard multipliers, u_n and u_{nt}. Unfortunately, these multipliers gave two identities (10) and (15), both of which involve nontrivial boundary integrals. Therefore, we need to understand the nature of these boundary integrals. In order to estimate these boundary integrals, we will need to prove the following two lemmas first. A similar result to Lemma 4 has been proven for the linear Schrödinger equation by Lasiecka and Triggiani in [6]. This is the nonlinear and inhomogeneous version of that result.

Lemma 4. *Let u_n be a solution of the problem* (7) *and $h \in [C^1(\overline{\Omega})]^n$ be a real vector field with the property $h|_\Gamma = \nu$, and let $H(x)$ be the $n\times n$ matrix with entries $H_{ij} = \frac{\partial h_i}{\partial x_j}$. Then, the following identity holds true*:

$$\begin{aligned}\frac{d}{dt}\operatorname{Im}(u_n, \nabla u_n\cdot h) &= \operatorname{Im}\langle Q_n, Q_{nt}\rangle + 2\operatorname{Re}(H\nabla u_n, \nabla u_n) + \|\nabla_A Q_n\|^2_\Gamma - \|\nabla u_n\cdot\nu\|^2_\Gamma \\ &\quad + \operatorname{Re}\big(\nabla(\operatorname{div} h)\cdot\nabla u_n, u_n\big) - \operatorname{Re}\langle\nabla u_n\cdot\nu, Q_n\operatorname{div} h\rangle + g\operatorname{Re}(B_n u_n, u_n\operatorname{div} h) \\ &\quad + 2g\operatorname{Re}(B_n u_n, \nabla u_n\cdot h) - 2\operatorname{Im}(au_n, \nabla u_n\cdot h).\end{aligned} \tag{16}$$

Proof. Let's define the functional

$$\rho_n := \operatorname{Im}(u_n, \nabla u_n\cdot h). \tag{17}$$

Differentiating ρ_n, we have

$$\rho_n' = \operatorname{Im}(u_{nt}, \nabla u_n\cdot h) + \operatorname{Im}(u_n, h\cdot\nabla u_{nt}). \tag{18}$$

Now, using the divergence theorem and the fact that $h|_\Gamma = \nu$, we have the equality

$$\operatorname{Im}(u_n, h\cdot\nabla u_{nt}) = \operatorname{Im}\langle u_n, u_{nt}\rangle - \operatorname{Im}(h\cdot\nabla u_n, u_{nt}) - \operatorname{Im}\big(\operatorname{div}(h)u_n, u_{nt}\big). \tag{19}$$

Now using (7) and the divergence theorem we get

$$\begin{aligned}-\operatorname{Im}\big(\operatorname{div}(h)u_n, u_{nt}\big) &= \big(\operatorname{div}(h)\nabla u_n, \nabla u_n\big) + \operatorname{Re}\big(\nabla\big(\operatorname{div}(h)\big)\cdot\nabla u_n, u_n\big) \\ &\quad + g\big(B_n u_n, u_n\operatorname{div}(h)\big) - \operatorname{Re}\big\langle\nabla u_n\cdot\nu, u_n\operatorname{div}(h)\big\rangle.\end{aligned} \tag{20}$$

We also have, by using (7),

$$-\operatorname{Im}(h\cdot\nabla u_n, u_{nt}) = -\operatorname{Re}(\Delta u_n, h\cdot\nabla u_n) + g\operatorname{Re}(B_n u_n, h\cdot\nabla u_n) + \operatorname{Re}(iau_n, h\cdot\nabla u_n), \tag{21}$$

where we compute

$$\begin{aligned}-\operatorname{Re}(\Delta u_n, h\cdot\nabla u_n) &= \operatorname{Re}\big(\nabla u_n, \nabla(h\cdot\nabla u_n)\big) - \|\nabla u_n\cdot\nu\|^2_\Gamma \\ &= \operatorname{Re}(H\nabla u_n, \nabla u_n) + \frac{1}{2}\big(h, \nabla\big(|\nabla u_n|^2\big)\big) - \|\nabla u_n\cdot\nu\|^2_\Gamma\end{aligned}$$

$$= \operatorname{Re}(H\nabla u_n, \nabla u_n) - \frac{1}{2}\big(\operatorname{div}(h)\nabla u_n, \nabla u_n\big) + \frac{1}{2}\|\nabla u_n\|_\Gamma^2 - \|\nabla u_n \cdot \nu\|_\Gamma^2. \tag{22}$$

Note that since w_n is constant on the boundary Γ, the tangential component of ∇w_n on the boundary is zero and we have that $\nabla w_n = \frac{\partial w_n}{\partial \nu}\nu$, i.e., ∇w_n is in the direction of the outward unit normal. Thus, using the definition of w_n, we have

$$\nabla u_n \cdot A = \overbrace{\nabla w_n \cdot A}^{0} + \nabla v_n \cdot A = \nabla v_n \cdot A$$

where A is the unit tangential vector, so the dot product with A gives the tangential components. Hence, we can write

$$|\nabla u_n|^2 = |\nabla u_n \cdot \nu|^2 + |\nabla u_n \cdot A|^2 = |\nabla u_n \cdot \nu|^2 + |\nabla_A Q_n|^2. \tag{23}$$

Using (17)–(23) and the boundary data $u_n|_\Gamma = Q_n$ we can rewrite ρ_n' as

$$\begin{aligned}\rho_n' = {} & \operatorname{Im}\langle Q_n, Q_{nt}\rangle + 2\operatorname{Re}(H\nabla u_n, \nabla u_n) + \|\nabla_A Q_n\|_\Gamma^2 - \|\nabla u_n \cdot \nu\|_\Gamma^2 \\ & + \operatorname{Re}\big(\nabla(\operatorname{div} h)\cdot \nabla u_n, u_n\big) - \operatorname{Re}\langle \nabla u_n \cdot \nu, Q_n \operatorname{div} h\rangle + g\operatorname{Re}(B_n u_n, u_n \operatorname{div} h) \\ & + 2g\operatorname{Re}(B_n u_n, \nabla u_n \cdot h) - 2\operatorname{Im}(a u_n, \nabla u_n \cdot h),\end{aligned}$$

which is the assertion of our lemma. □

Before using Lemma 4 to attempt to get some estimates, we will need to prove another lemma given below which estimates the term $2g\operatorname{Re}(B_n u_n, \nabla u_n \cdot h)$ that appears in identity (16). Note that computation of this term for the operator B instead of B_n is easy, but we need to be more careful to get a similar estimate for B_n.

Lemma 5. *Let u_n and h be as in Lemma 4. Then,*

$$\operatorname{Re}(B_n u_n, u_n \operatorname{div} h) + 2\operatorname{Re}(B_n u_n, \nabla u_n \cdot h) \leqslant M_1 \|Q_n\|_{L^{p+2}(\Gamma)}^{p+2} + M_2 \psi_n(u_n)$$

where M_1, M_2 are positive constants.

Proof. Consider the map $b : \mathbb{C} \to \mathbb{C}$ defined by $b(z) = |z|^p z$ and the corresponding Yosida approximations b_n of b with the respective subdifferentials $\phi : \mathbb{C} \to \mathbb{R}_\infty$ and $\phi_n : \mathbb{C} \to \mathbb{R}_\infty$ where $\partial\phi = b$ and $\partial\phi_n = b_n$. Note that these are essentially the same operators and corresponding subdifferentials given in the main section, except that we are considering them on $\mathbb{C}$ instead of H for a moment. Now, let $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by $\mu(r) = \frac{1}{p+2} r^{\frac{p+2}{2}} + \frac{1}{2n} r^{p+1}$. Using this function we can rewrite ϕ as

$$\phi_n(z) = \mu\big(\big|j_n(z)\big|^2\big).$$

Therefore, we have the identity

$$\begin{aligned}\operatorname{div}\big(\phi_n\big(u_n(x)\big)h(x)\big) = {} & 2\operatorname{Re}\big(\mu'\big(\big|j_n u_n(x)\big|^2\big) j_n u_n(x) \nabla_x \overline{j_n u_n}(x) \cdot h(x)\big) \\ & + \phi_n\big(u_n(x)\big)\operatorname{div}\big(h(x)\big).\end{aligned} \tag{24}$$

Integrating the first term on the right-hand side of (24), we get

$$\begin{aligned} 2\,\mathrm{Re}\big(\mu'\big(|J_n u_n|^2\big) J_n u_n, \nabla J_n u_n \cdot h\big) &= 2\,\mathrm{Re}(B_n u_n, \nabla J_n u_n \cdot h) \\ &= 2\,\mathrm{Re}\left(B_n u_n, \nabla\left(u_n - \frac{1}{n} B_n u_n\right)\cdot h\right) \\ &= 2\,\mathrm{Re}(B_n u_n, \nabla u_n \cdot h) - \frac{2}{n}\mathrm{Re}(B_n u_n, \nabla B_n u_n \cdot h). \end{aligned}$$

Therefore,

$$2\,\mathrm{Re}(B_n u_n, \nabla u_n \cdot h) = \int_\Omega \mathrm{div}\big(\phi_n(u_n)h\big)\,dx - \int_\Omega \phi_n(u_n)\,\mathrm{div}(h)\,dx + \frac{2}{n}\mathrm{Re}(B_n u_n, \nabla B_n u_n \cdot h). \tag{25}$$

Using the divergence theorem on the first and third terms of the right-hand side of (25) and using the fact that $\phi_n(z) \leqslant \phi(z)$ and $u_n|_\Gamma = Q_n$ we get

$$\int_\Omega \mathrm{div}\big(\phi_n(u_n)h\big)\,dx = \int_\Gamma \phi_n(u_n) h\cdot\nu\,d\Gamma \leqslant \frac{1}{p+2}\|Q_n\|^{p+2}_{L^{p+2}(\Omega)} \tag{26}$$

and

$$\begin{aligned} \mathrm{Re}(B_n u_n, \nabla B_n u_n \cdot h) &= \mathrm{Re}\int_\Gamma |B_n u_n|^2 h\cdot\nu\,d\Gamma - \mathrm{Re}\big(B_n u_n, \mathrm{div}(B_n u_n h)\big) \\ &= \mathrm{Re}\int_\Gamma |B_n u_n|^2 h\cdot\nu\,d\Gamma - \mathrm{Re}(B_n u_n, \nabla B_n u_n \cdot h) \\ &\quad - \mathrm{Re}\big(B_n u_n, B_n u_n \cdot \mathrm{div}(h)\big) \end{aligned} \tag{27}$$

which implies

$$\begin{aligned} \frac{2}{n}\mathrm{Re}(B_n u_n, \nabla B_n u_n \cdot h) &\leqslant \frac{1}{n}\|B_n u_n\|^2_\Gamma + \frac{M}{n}\|B_n u_n\|^2 \leqslant 2\psi_n(Q_n) + 2M\psi_n(u_n) \\ &\leqslant \frac{2}{p+2}\|Q_n\|^{p+2}_{L^{p+2}(\Omega)} + 2M\psi_n(u_n), \end{aligned} \tag{28}$$

where $M = \max_\Omega|\mathrm{div}(h)|$.

On the other hand, with a calculation similar to (9), we have

$$\mathrm{Re}(B_n u_n, u_n\,\mathrm{div}\,h) \leqslant M\big(p\psi(J_n u_n) + \psi_n(u_n)\big) \leqslant M_3\psi_n(u_n). \tag{29}$$

Now, combining (25)–(29), we get the final estimate

$$\mathrm{Re}(B_n u_n, u_n\,\mathrm{div}\,h) + 2\,\mathrm{Re}(B_n u_n, \nabla u_n \cdot h) \leqslant M_1\|Q_n\|^{p+2}_{L^{p+2}(\Gamma)} + M_2\psi_n(u_n),$$

where M_1, M_2 are positive constants. □

Let's define

$$G_n := \|u_n\|^2_{H^1(\Omega)} + 2g\psi_n(u_n) \quad \text{and} \quad G_{n,\epsilon} := G_n + \epsilon\rho_n,$$

where ϵ is a fixed positive constant to be chosen later. Using (10) and (15) together with the definition of b-norm of boundary data and ϵ-Young's inequality we have

$$G_n' \leqslant -2aG_n + \frac{C_0}{\epsilon}\|Q_n\|_w^2 + \frac{\epsilon}{2}\|\nabla u_n \cdot \nu\|_\Gamma^2$$

for some $C_0 > 0$. By definition of ρ_n, we have

$$|\rho_n| \leqslant C_1 G_n$$

for some $C_1 > 0$. Using Lemmas 4 and 5, we write an estimate for ρ_n' as

$$\rho_n' \leqslant C_2 G_n + C_3\|Q_n\|_w^2 - \frac{1}{2}\|\nabla u \cdot \nu\|_\Gamma^2.$$

Hence,

$$G_n' + \epsilon\rho_n' = G_{n,\epsilon}' \leqslant (C_2\epsilon - 2a)G_n + \left(C_3\epsilon + \frac{C_0}{\epsilon}\right)\|Q_n\|_w^2.$$

Choosing ϵ sufficiently small, we have

$$G_{n,\epsilon}' \leqslant (C_2\epsilon - 2a)G_n + \left(C_3\epsilon + \frac{C_0}{\epsilon}\right)\|Q_n\|_w^2.$$

Since $|\rho_n| \leqslant C_1 G_n$, we have

$$G_{n,\epsilon} = G_n + \epsilon\rho_n \leqslant G_n + \epsilon C_1 G_n = (1 + \epsilon C_1)G_n$$

which implies

$$\frac{C_2\epsilon - 2a}{1 + \epsilon C_1} G_{n,\epsilon} \geqslant (C_2\epsilon - 2a)G_n$$

for sufficiently small ϵ.

Hence,

$$G_{n,\epsilon}' \leqslant \frac{C_2\epsilon - 2a}{1 + \epsilon C_1} G_{n,\epsilon} + \left(C_3\epsilon + \frac{C_0}{\epsilon}\right)\|Q_n\|_w^2.$$

After integration in time, we get

$$G_{n,\epsilon}(t) \leqslant G_{n,\epsilon}(0)e^{\frac{C_2\epsilon - 2a}{1+\epsilon C_1}t} + \left(C_3\epsilon + \frac{C_0}{\epsilon}\right)e^{\frac{C_2\epsilon - 2a}{1+\epsilon C_1}t}\int_0^t e^{\frac{2a - C_2\epsilon}{1+\epsilon C_1}s}\left\|Q_n(s)\right\|_w^2 ds.$$

Replacing $G_{n,\epsilon}$ with $G_n + \epsilon\rho_n$ and using $|\rho_n| \leqslant C_1 G_n$, we get

$$(1-\epsilon C_1)G_n(t) \leqslant (1+\epsilon C_1)G_n(0)e^{\frac{C_2\epsilon-2a}{1+\epsilon C_1}t} + \left(C_3\epsilon + \frac{C_0}{\epsilon}\right)e^{\frac{C_2\epsilon-2a}{1+\epsilon C_1}t}\int_0^t e^{\frac{2a-C_2\epsilon}{1+\epsilon C_1}s}\left\|Q_n(s)\right\|_w^2\,ds$$

which gives

$$G_n(t) \leqslant \frac{(1+\epsilon C_1)}{(1-\epsilon C_1)}G_n(0)e^{\frac{C_2\epsilon-2a}{1+\epsilon C_1}t} + \frac{(C_3\epsilon+\frac{C_0}{\epsilon})}{(1-\epsilon C_1)}e^{\frac{C_2\epsilon-2a}{1+\epsilon C_1}t}\int_0^t e^{\frac{2a-C_2\epsilon}{1+\epsilon C_1}s}\left\|Q_n(s)\right\|_w^2\,ds. \tag{30}$$

From the estimate (30), using definition of G_n and the fact that $\psi_n(u_n(0)) = \psi_n(u_{n0}) \leqslant \psi(u_{n0})$, and choosing ϵ small enough, we can conclude the following lemma.

Lemma 6. *Let u_n be the unique solution of* (7)*, then for any $b < a$, there exists some constant $C > 0$ such that*

$$\|u_n\|_{H^1(\Omega)}^2 + \|J_n u_n\|_{L^{p+2}(\Omega)}^{p+2} \leqslant Ce^{-2bt}\left(\|u_0\|_{H^1(\Omega)\cap L^{p+2}(\Omega)}^2 + \int_0^t e^{2bs}\left\|Q(s)\right\|_w^2\,ds\right).$$

The first consequence of this estimate is that u_n can be extended globally and the sequences $\{u_n\}$ and $\{J_n u_n\}$ are bounded in the spaces $L^2(0,T;H^1(\Omega))$ and $L^{p+2}(0,T;L^{p+2}(\Omega))$, respectively. It follows by duality that Δu_n and $B(J_n u_n)$ are bounded in the spaces $L^2(0,T;H^{-1}(\Omega))$ and $L^{(p+2)'}(0,T;L^{(p+2)'}(\Omega))$. We know that $H^k(\Omega) \hookrightarrow H^1(\Omega) \cap L^{p+2}(\Omega)$, where $k > n/2$. Hence, $\{u_n\}$ is bounded in the space

$$X := \left\{u \in L^2\big(0,T;H^1(\Omega)\big)\colon u' \in L^{(p+2)'}\big(0,T;\big(H^k(\Omega)\big)'\big)\right\}.$$

On the other hand, $H^1(\Omega) \Subset L^2(\Omega) \hookrightarrow (H^k(\Omega))'$. Then, we have $X \Subset L^2(U) = L^2(0,T;L^2(\Omega))$ by Lions–Aubin Lemma. Thus there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in L^2(U)$ such that

$$u_{n_k} \to u \quad \text{in } L^2(U) \text{ and a.e. on } U.$$

Since the operator B considered on $\mathbb{C}$ is also accretive, we have the following estimate:

$$|J_{n_k}u_{n_k} - u| \leqslant |J_{n_k}u_{n_k} - J_{n_k}u| + \frac{1}{n_k}|B_{n_k}u_{n_k}| \leqslant |u_{n_k} - u| + \frac{1}{n_k}|Bu|.$$

Letting $k \to \infty$, we obtain $J_{n_k}u_{n_k} \to u$ and $B_{n_k}u_{n_k} \to Bu$ a.e. on U, but this implies

$$Ju_{n_k}u_{n_k} \rightharpoonup u \quad \text{in } L^{p+2}(U) = L^{p+2}\big(0,T;L^{p+2}(\Omega)\big),$$
$$B_{n_k}u_{n_k} \rightharpoonup |u|^p u \quad \text{in } L^{(p+2)'}(U) = L^{(p+2)'}\big(0,T;L^{(p+2)'}(\Omega)\big).$$

Furthermore, we have

$$u_{n_k} \rightharpoonup u \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),$$

and by Lemma 6, it also follows that

$$u \in L^{\infty}\big(0, T; H^1(\Omega) \cap L^{p+2}(\Omega)\big).$$

Now, let $\varphi \in C^1(0, T; H_0^1(\Omega) \cap L^{p+2}(\Omega))$ with $\phi(T) = 0$. Then, from (7), we have

$$\int_0^T -(u_{n_k}, \varphi_t) + i(\nabla u_{n_k}, \nabla \varphi) + i[B_{n_k} u_{n_k}, \varphi]_{L^{(p+2)'}, L^{(p+2)}} - a(u_{n_k}, \varphi)\, dt = \big(u_0, \varphi(0)\big).$$

Passing to the limit as $k \to \infty$, we have

$$\int_0^T -(u, \varphi_t) + i(\nabla u, \nabla \varphi) + i\big[|u|^p u, \varphi\big]_{L^{(p+2)'}, L^{(p+2)}} - a(u, \varphi)\, dt = \big(u_0, \varphi(0)\big). \tag{31}$$

Moreover, the trace of u on Γ has to be equal to the boundary data Q. To see this we write $w_{n_k} = u_{n_k} - v_{n_k} \in L^2(0, T; H_0^1(\Omega))$, which we know weakly converges to $u - v$ in $L^2(0, T; H^1(\Omega))$. Therefore we have $u - v \in L^2(0, T; H_0^1(\Omega))$ since $L^2(0, T; H_0^1(\Omega))$ is a closed and convex, and therefore weakly closed subset of $L^2(0, T; H^1(\Omega))$. Hence, $u|_\Gamma = v|_\Gamma = Q$ in $L^2(0, T; H^{1/2}(\Gamma))$. In addition, from identity (31), it follows that u belongs to the space

$$X := \big\{u \in L^2\big(0, T; H^1(\Omega) \cap L^{p+2}(\Omega)\big) \text{ such that } u' \in L^2\big(0, T; H^{-1}(\Omega) + L^{(p+2)'}(\Omega)\big)\big\},$$

which can be continuously embedded in the space $C([0, T]; L^2(\Omega))$. This is a particular case of the result that is proved in Showalter [11, Chapter III, Proposition 1.2]. Hence, we just showed that all the requirements of the definition of the weak solution of problem (1)–(2) are satisfied. The stabilization result follows by applying the fact that "*any uniform bound for the weakly convergent sequence becomes also a bound for its weak limit*" to the subsequence $\{u_{n_k}\}$. Hence, for any $b < a$ there is some $C > 0$ such that

$$\big\|u(t)\big\|^2_{H^1(\Omega)\cap L^{p+2}(\Omega)} \leqslant Ce^{-2bt}\left(\|u_0\|^2_{H^1(\Omega)\cap L^{p+2}(\Omega)} + \int_0^t e^{2bs}\big\|Q(s)\big\|_w^2\, ds\right). \tag{32}$$

This implies

$$\lim_{t\to\infty}\big\|u(t)\big\|_{H^1(\Omega)\cap L^{p+2}(\Omega)} = 0$$

due to the fact that $\|Q(s)\|_w \to 0$. Note that from Eq. (32), it follows that the rate of decay is at least

$$\max\big\{O\big(e^{-2bt}\big), O\big(\big\|Q(t)\big\|_w\big)\big\}. \qquad \square$$

5. Proof of Theorem 2 – local regular solutions

This section is devoted to the proof of Theorem 2. To this end we shall use a different approximation of nonlinear term than the one used for weak solutions. In fact, the approximation used below relies on truncation, rather than exploring monotonicity. In that respect, it is somewhat related to approximations used in [14] in the context of weak solutions.

Proof. In order to obtain strong solutions for the nonlinear problem, we again extend the inhomogeneous boundary data as a strong solution of the corresponding linear equation, and then homogenize the nonlinear equation. Indeed, we will use the following linear result stated below in Theorem 5.

Theorem 5. *The problem*

$$\begin{cases} i\dfrac{dv}{dt} - \Delta v = 0 & \text{in } \Omega_T, \\ v = Q & \text{on } \Gamma_T, \\ v(0) = u_0 & \text{in } \Omega \end{cases} \tag{33}$$

with control function $Q \in H^{2,2}(\Gamma_T)$ satisfying the compatibility condition $Q(0) = u_0|_\Gamma$, initial data $u_0 \in H^2(\Omega)$, possesses a unique solution $v \in C([0,T]; H^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ together with $\frac{\partial v}{\partial \nu} \in H^1(\Gamma_T)$.

Remark 9. (i) Solutions described by Theorem 5 also satisfy the continuous dependence on data. (ii) Lemma 1, Theorems 4 and 5 can be easily extended to higher regularity levels. Therefore, it is possible to get corresponding results also at higher regularity levels. The regularity of the local solutions needed for integration by parts in the next section is also justified with this fact.

Theorem 5 follows easily from Theorem 4 by mimicking exactly the same functional analysis based arguments given in the paper of Lasiecka, Lions and Triggiani [4], which is originally presented for the wave equation.

Now, using Theorem 5, we can extend the boundary control Q as a solution $v \in C([0,T]; H^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of the linear Schrödinger equation given in (33).

We define $w := u - v$, and we see that instead of directly working on (1)–(2), it is enough to solve the problem for w given as

$$\begin{cases} i\dfrac{dw}{dt} - \Delta w + F_v(w) = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \Gamma \times (0,T), \\ w(0) = 0 & \text{in } \Omega, \end{cases}$$

where $F_v(w) = g|w+v|^p(w+v) + ia(w+v)$.

We will consider the w-problem first with the following truncated nonlinearity

$$F_{\delta,v}(w) := \begin{cases} |w+v|^p(w+v) + ia(w+v) & \text{for } |w+v| \leqslant \delta, \\ \delta^{p+1}\exp(i\arg((w+v)(x))) + ia(w+v) & \text{otherwise,} \end{cases}$$

which is Lipschitz on $L^2(\Omega)$.

Therefore, by Lemma 1, there exists a unique strong solution $w_\delta \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to the problem

$$\begin{cases} \dfrac{dw}{dt} + i\Delta w - iF_{\delta,v}(w) = 0 & \text{on } (0,T), \\ w(0) = 0 \end{cases}$$

for all $\delta > 0$.

For each $\delta > 0$, writing $u_\delta := w_\delta + v$, we see that $u_\delta \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega))$ is a unique solution for the problem

$$\begin{cases} \dfrac{du}{dt} + i\Delta u - iF_\delta(u) = 0 & \text{on } (0,T), \\ u(0) = u_0, \end{cases} \tag{34}$$

where

$$F_\delta(u) := \begin{cases} |u|^p u + iau & \text{for } |u| \leqslant \delta, \\ \delta^{p+1}\exp(i\arg(u(x))) + iau & \text{otherwise.} \end{cases}$$

Moreover, u_δ can be extended globally.

Now, let $\delta = 1 + \|u_0\|_L^\infty$ be a fixed number. Then by the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and continuity of u_δ there exists $T' > 0$ such that $\|u_\delta\|_{L^\infty(\Omega)} < \delta$ on $[0, T']$, and hence

$$F_\delta(u_\delta) = |u_\delta|^p u_\delta + iau_\delta$$

on $[0, T']$. In other words, $u_\delta \in C^1([0, T']; L^2(\Omega)) \cap C([0, T']; H^2(\Omega))$ is the unique solution of the problem (5) on the interval $[0, T']$. This proves the local existence of strong solutions.

Now, let $T_{\max}$ be the maximal time such that u-problem has a strong solution u on $[0, T_{\max})$.

We want to prove that if $T_{\max} < \infty$, then $\limsup_{t\to T_{\max}} \|u\|_{L^\infty(\Omega)} = \infty$. Let us suppose to the contrary that in the case $T_{\max} < \infty$, $\limsup_{t\to T_{\max}} \|u\|_{L^\infty(\Omega)} =: \delta_\infty < \infty$. Then, u solves (34) with $\delta = \delta_\infty$, which implies $u = u_{\delta_\infty}$ on $[0, T_{\max})$ by uniqueness. Since we know that $u_{\delta_\infty}(T_{\max}) \in H^2(\Omega)$, the following problem is well defined:

$$\begin{cases} i\dfrac{du}{dt} - \Delta u + F_\epsilon(u) = 0 \quad \text{on } (T_{\max}, T_{\delta_1}), \\ u(T_{\max}) = u_{\delta_\infty}(T_{\max}). \end{cases} \tag{35}$$

Let $\epsilon := \delta_\infty + 1$. Then by the same continuity argument, there exists a time $T'' > T_{\max}$ such that the solution u_ϵ of (35) satisfies $\|u_\epsilon\|_{L^\infty(\Omega)} \leqslant \epsilon$ on $[T_{\max}, T'']$. Therefore,

$$\tilde{u} := \begin{cases} u & \text{on } [0, T_{\max}), \\ u_\epsilon & \text{on } [T_{\max}, T'') \end{cases}$$

is an extension of u that solves (1)–(2). However, this contradicts the definition of $T_{\max}$. Hence, we conclude that if $T_{\max} < \infty$, then $\max_{t\to T_{\max}} \|u\|_{L^\infty(\Omega)} = \infty$. $\square$

6. Proof of Theorem 3 – global and decaying regular solutions

This section is devoted to the proof of Theorem 3. We will again use the direct multiplier and perturbed energy techniques adapted to H^2-topology.

Proof. Taking the Laplacian of the main equation, we obtain

$$\Delta u_t + i\Delta^2 u + a\Delta u = ig|u|^p\delta u + ig\frac{p}{2}|u|^{p-2}\nabla u\cdot\nabla|u|^2 + ig\frac{p(p-2)}{4}u|u|^{p-4}\left(\nabla|u|^2\right)^2 + ig\frac{p}{2}u|u|^{p-2}\Delta|u|^2. \tag{36}$$

Now, taking the inner product of (36) with $\Delta\bar{u}$ and looking at real parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + a\|\Delta u\|^2 = \operatorname{Im}\left\langle\frac{\partial\Delta u}{\partial\nu}, \Delta u\right\rangle - \operatorname{Re}\left(\frac{p}{2}|u|^{p-2}\nabla u\cdot\nabla|u|^2, \Delta u\right) - \operatorname{Re}\left(\frac{p(p-2)}{4}u|u|^{p-4}\left(\nabla|u|^2\right)^2, \Delta u\right) - \operatorname{Re}\left(\frac{p}{2}u|u|^{p-2}\Delta|u|^2, \Delta u\right). \tag{37}$$

The first term at the right-hand side of (37) can be estimated as follows:

$$\begin{aligned}\operatorname{Im}\left\langle \frac{\partial \Delta u}{\partial \nu}, \Delta u\right\rangle &= \operatorname{Im}\left\langle \frac{\partial (iu_t + g|u|^p u + iau)}{\partial \nu}, \tilde{Q}\right\rangle = -\operatorname{Re}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}_t\right\rangle + \frac{d}{dt}\operatorname{Im}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle \\ &\quad + \operatorname{Im}\left\langle \frac{p+2}{2}|u|^p\frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle + \operatorname{Im}\left\langle \frac{p}{2}|u|^{p-2}u^2\frac{\partial \bar{u}}{\partial \nu}, \tilde{Q}\right\rangle + \operatorname{Im}\left\langle ia\frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle \\ &\leqslant \frac{\epsilon}{4}\left\|\frac{\partial u}{\partial \nu}\right\|^2 + \frac{C}{\epsilon}\|Q\|_s^2 + \frac{d}{dt}\operatorname{Im}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle \end{aligned} \tag{38}$$

where

$$\tilde{Q} = iQ_t + g|Q|^pQ + iaQ$$

and

$$\|Q(t)\|_s^2 := \|Q(t)\|_{H^2(\Gamma)}^2 + \|Q_t\|_{L^2(\Gamma)}^2 + \|Q_{tt}\|_{L^2(\Gamma)}^2. \tag{39}$$

We know that $|\nabla|u|^2| \leqslant 2|\nabla u||u|$, therefore the second and the third terms at the right-hand side of (37) can be estimated as follows:

$$\begin{aligned} -\operatorname{Re}\left(\frac{p}{2}|u|^{p-2}\nabla u\cdot\nabla|u|^2, \Delta u\right) &\leqslant p\int_\Omega |u|^{p-1}|\nabla u|^2|\Delta u|\,dx \\ &\leqslant p\|u\|_{L^\infty(\Omega)}^{p-1}\|\nabla u\|_{L^4(\Omega)}^2\|\Delta u\| \end{aligned} \tag{40}$$

and

$$\begin{aligned} -\operatorname{Re}\left(\frac{p(p-2)}{4}u|u|^{p-4}\left(\nabla|u|^2\right)^2, \Delta u\right) &\leqslant \frac{p|p-2|}{2}\int_\Omega |u|^{p-1}|\nabla u|^2|\Delta u|\,dx \\ &\leqslant \frac{p|p-2|}{2}\|u\|_{L^\infty(\Omega)}^{p-1}\|\nabla u\|_{L^4(\Omega)}^2\|\Delta u\|. \end{aligned} \tag{41}$$

We note that

$$\left|\Delta|u|^2\right| \leqslant 2|\Delta u||u| + 2|\nabla u|^2.$$

Therefore the last term at the right-hand side of (37) can be estimated as follows:

$$\begin{aligned} -\operatorname{Re}\left(\frac{p}{2}u|u|^{p-2}\Delta|u|^2, \Delta u\right) &\leqslant \frac{p}{2}\|u\|_{L^\infty(\Omega)}^{p-1}\int_\Omega \left(2|\Delta u||u| + 2|\nabla u|^2\right)|\Delta u|\,dx \\ &\leqslant p\|u\|_{L^\infty(\Omega)}^{p}\|\Delta u\|^2 + \frac{p}{2}\|u\|_{L^\infty(\Omega)}^{p-1}\|\nabla u\|_{L^4(\Omega)}^2\|\Delta u\|. \end{aligned} \tag{42}$$

Using the estimates in (38) and (40)–(42), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 \leqslant -a\|\Delta u\|^2 + C\|u\|_{L^\infty(\Omega)}^{p-1}\|\nabla u\|_{L^4(\Omega)}^2\|\Delta u\| + C\|u\|_{L^\infty(\Omega)}^{p}\|\Delta u\|^2 + \frac{\epsilon}{4}\left\|\frac{\partial u}{\partial \nu}\right\|^2 + \frac{C}{\epsilon}\|Q\|_s^2 + \frac{d}{dt}\operatorname{Im}\left\langle\frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle.$$

We already know by Theorem 1 that for any $b < a$, there is some $C > 0$ such that

$$\|u\|_{H^1(\Omega)\cap L^{p+2}(\Omega)} \leqslant C\Upsilon(t)e^{-bt},$$

where

$$\Upsilon(t) := \left(\|u_0\|_{H^1(\Omega)\cap L^{p+2}(\Omega)}^2 + \int_0^t e^{2as}\|Q(s)\|_w^2\,ds\right)^{1/2}.$$

Hence, we have

$$C\|u\|_{L^\infty(\Omega)}^{p-1}\|\nabla u\|_{L^4(\Omega)}^2\|\Delta u\| + C\|u\|_{L^\infty(\Omega)}^{p}\|\Delta u\|^2 \leqslant C\|\Delta u\|^{\frac{pn}{4}+2}\Upsilon(t)^{p(1-n/4)}e^{-(1-n/4)pbt} + C\Upsilon(t)^{p+2}e^{-(p+2)bt}.$$

Let's define

$$G := \|u\|_{H^2(\Omega)}^2 + \frac{2g}{p+2}\|u\|_{p+2}^{p+2} \quad \text{and} \quad G_\epsilon := G + \epsilon\rho,$$

where $\rho = \operatorname{Im}(u, \frac{\partial u}{\partial \nu})$. Then, we have

$$G' \leqslant -2aG + C\|\Delta u\|^{\frac{pn}{4}+2}\Upsilon(t)^{p(1-n/4)}e^{-(1-n/4)p\frac{C_6}{2}t} + C\Upsilon(t)^{p+2}e^{-(p+2)\frac{C_6}{2}t} + \frac{C_0}{\epsilon}\|Q\|_s^2 + \frac{3\epsilon}{4}\|\nabla u\cdot\nu\|_\Gamma^2 + \frac{d}{dt}\left(\operatorname{Im}\left\langle\frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle\right)$$

for some $C_0 > 0$. By definition of ρ, we have

$$|\rho| \leqslant C_1 G$$

for some $C_1 > 0$. Using Lemmas 4 and 5, we can estimate ρ' as

$$\rho' \leqslant C_2 G + C_3\|Q\|_s^2 - \frac{7}{8}\|\nabla u\cdot\nu\|_\Gamma^2.$$

Hence,

$$G' + \epsilon\rho' = G'_\epsilon \leqslant (C_2\epsilon - 2a)G - \frac{\epsilon}{8}\|\nabla u\cdot\nu\|_\Gamma^2 + \left(C_3\epsilon + \frac{C_0}{\epsilon}\right)\|Q\|_s^2 + C\|\Delta u\|^{\frac{pn}{4}+2}\Upsilon(t)^{p(1-n/4)}e^{-(1-n/4)p\frac{C_6}{2}t} + C\Upsilon(t)^{p+2}e^{-(p+2)\frac{C_6}{2}t} + \frac{d}{dt}\left(\operatorname{Im}\left\langle\frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle\right).$$

Choosing ϵ sufficiently small, we have

$$G'_\epsilon \leqslant -C_5 G + C_4 \|Q\|_s^2 + C\|\Delta u\|^{\frac{pn}{4}+2} \Upsilon(t)^{p(1-n/4)} e^{-(1-n/4)p\frac{C_6}{2}t} + C\Upsilon(t)^{p+2} e^{-(p+2)\frac{C_6}{2}t} + \frac{d}{dt}\left(\operatorname{Im}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle\right),$$

where $C_4 := C_3\epsilon + \frac{C_0}{\epsilon}$, $C_5 := 2a - C_2\epsilon$. Since $|\rho| \leqslant C_1 G$, we have

$$G_\epsilon = G + \epsilon\rho \leqslant G + \epsilon C_1 G = (1 + \epsilon C_1)G,$$

which implies

$$\frac{-C_5}{1+\epsilon C_1} G_\epsilon \geqslant -C_5 G.$$

Hence,

$$G'_\epsilon \leqslant -C_6 G_\epsilon + C_4 \|Q\|_s^2 + C\|\Delta u\|^{\frac{pn}{4}+2} \Upsilon(t)^{p(1-n/4)} e^{-(1-n/4)p\frac{C_6}{2}t} + C\Upsilon(t)^{p+2} e^{-(p+2)\frac{C_6}{2}t} + \frac{d}{dt}\left(\operatorname{Im}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle\right),$$

where $C_6 := \frac{C_5}{1+\epsilon C_1}$.

Multiplying the inequality with $e^{C_6 t}$, one gets

$$\frac{d}{dt}\left(G_\epsilon e^{C_6 t}\right) \leqslant C_4 e^{C_6 t} \|Q\|_s^2 + C\|\Delta u\|^{\frac{pn}{4}+2} e^{C_6 t} \Upsilon(t)^{p(1-n/4)} e^{-(1-n/4)p\frac{C_6}{2}t} + C\Upsilon(t)^{p+2} e^{-(p+2)\frac{C_6}{2}t} e^{C_6 t} + \frac{d}{dt}\left(\operatorname{Im}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle\right) e^{C_6 t}.$$

By integrating the last inequality, we obtain

$$\begin{aligned}(1-\epsilon C_1)G(t)e^{C_6 t} \leqslant{}& (1+\epsilon C_1)G(0) + C_4 \int_0^t e^{C_6 s}\|Q(s)\|_s^2\,ds \\ &+ C\int_0^t \|\Delta u\|^{\frac{pn}{4}+2} e^{C_6 s}\Upsilon(s)^{p(1-n/4)} e^{-(1-n/4)p\frac{C_6}{2}s}\,ds \\ &+ C\int_0^t \Upsilon(s)^{p+2} e^{-(p+2)\frac{C_6}{2}s} e^{C_6 s}\,ds + \operatorname{Im}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle e^{C_6 t} \\ &- C_6 \int_0^t \operatorname{Im}\left\langle \frac{\partial u}{\partial \nu}, \tilde{Q}\right\rangle e^{C_6 s}\,ds.\end{aligned}$$

Hence, we have

$$M(t) \leqslant K_1 G(0)^2 + K_2 \left\| Q(t) \right\|_s^2 + K_3 \int_0^t \left\| Q(s) \right\|_s^2 e^{C_6 s}\, ds + \epsilon \int_0^t M(s)\, ds$$
$$+ K_5 \int_0^t \Upsilon(s)^{p(1-n/4)} e^{-p\frac{C_6}{2}s} M(s)^{(\frac{pn}{4}+2)/2}\, ds + K_6 \int_0^t \Upsilon(s)^{p+2} e^{-ps}\, ds,$$

where $M(t) := G(t)e^{C_6 t}$.

Now, we appeal to the following lemma, see Perov [7].

Lemma 7. *Let $\alpha > 1$, $M(t), D(t)$, and $F(t)$ be nonnegative continuous functions on $[0, t_{\max})$ such that the inequality*

$$M(t) \leqslant C + \int_0^t \varphi_1(s) M(s)\, ds + \int_0^t \varphi_2(s) M(s)^\alpha\, ds$$

holds true. Then

$$M(t) \leqslant e^{\int_0^t \varphi_1(s)\, ds} \left(C^{1-\alpha} + (1-\alpha) \int_0^t \varphi_2(s) e^{(\alpha-1)\int_0^s \varphi_1(\xi)\, d\xi}\, ds \right)^{\frac{1}{1-\alpha}}$$

provided that

$$C^{1-\alpha} + (1-\alpha) \int_0^t \varphi_2(s) e^{(\alpha-1)\int_0^s \varphi_1(\xi)\, d\xi}\, ds \geqslant 0$$

for any $t \in [0, t_{\max})$.

By Lemma 7 we conclude that for any $b < a$, there exists a positive number C such that

$$\|u\|_{H^2(\Omega)} \leqslant G(t)^{\frac{1}{2}} \leqslant Ce^{-bt} \tag{43}$$

provided that initial data and boundary control are small enough. More precisely, we require that there exist $\delta, C > 0$ with

$$K_1 G(0)^2 + K_2 \left\| Q(t) \right\|_s^2 + K_3 \int_0^t \left\| Q(s) \right\|_s^2 e^{C_6 s}\, ds + K_6 \int_0^t \Upsilon(s)^{p+2} e^{-ps}\, ds < C \tag{44}$$

and

$$\int_0^t \Upsilon(s)^{p(1-n/4)} e^{(-p\frac{C_6}{2} + \epsilon\frac{np}{8})s}\, ds \leqslant \frac{8(C^{\frac{pn}{8}} - \delta)}{np}. \tag{45}$$

Due to the Sobolev embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ in dimensions $n = 1, 2, 3$, the stabilization estimate (43) also proves that $\limsup_{t\to T_{\max}} \|u\|_{L^\infty(\Omega)}$ is bounded. Thus, the local strong solution is indeed global and decays to zero. □

Remark 10. We note that the decay rates imposed on the boundary data are almost optimal. Indeed, the constant C_6 in (44) can be made arbitrarily close to $2a$ which is optimal rate of decay for the energy of the corresponding homogeneous boundary value problem.

Remark 11. In the definition of s-norm given in (39) for the boundary control, we assume for simplicity that $\partial\Omega$ is smooth enough (say $(n-1)$-dimensional smooth manifold) so that Sobolev embeddings hold true on the trace space. Otherwise, one also needs to add relevant L^q-norms to the definition (39) and change the assumption on Q in the theorem accordingly.

References

[1] C. Bu, K. Tsutaya, C. Zhang, Nonlinear Schrödinger equation with inhomogeneous Dirichlet boundary data, J. Math. Phys. 46 (8) (2005).
[2] L. Deng, P. Yiao, Boundary controllability for the semilinear Schrödinger equations on Riemannian manifolds, J. Math. Anal. Appl. 272 (2010) 19–44.
[3] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, 1983.
[4] I. Lasiecka, J.-L. Lions, R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures Appl. (9) 65 (2) (1986) 149–192.
[5] I. Lasiecka, R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories. II, Cambridge University Press, 2000.
[6] I. Lasiecka, R. Triggiani, Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control, Differential Integral Equations 5 (3) (1992) 521–535.
[7] N.I. Perov, K voprosu o strukture integral'noi voronki, Nauchn. Dokl. Vyssh. Shkoly Ser. FMN(2) (1959).
[8] M. Tsutsumi, On global solutions to the initial boundary value problem for the damped nonlinear Schrödinger equations, J. Math. Anal. Appl. 145 (2) (1990) 328–341.
[9] N. Okazawa, T. Yokota, Monotonicity method applied to the complex Ginzburg–Landau and related equations, J. Math. Anal. Appl. 267 (2) (2002) 247–263.
[10] N. Okazawa, T. Yokota, Perturbations of maximal monotone operators applied to the nonlinear Schrödinger and complex Ginzburg–Landau equations, in: Nonlinear Evolution Equations and Applications, Kyoto, 1998, in: Surikaisekikenkyusho Kokyuroku, vol. 1105, 1999, 102DEC:150, HEX:96120 (in Japanese).
[11] R.E. Showalter, Monotone Operators in Banach Space and Nonlinear Differential Equations, Math. Surveys Monogr., vol. 49, Amer. Math. Soc., Providence, RI, 1997.
[12] L. Rosier, B.-Y. Zhang, Local exact controllability and stabilizability of the nonlinear Schrödinger equation on a bounded interval, SIAM J. Control Optim. 48 (2) (2009).
[13] L. Rosier, B.-Y. Zhang, Exact boundary controllability of the nonlinear Schrödinger equation, J. Differential Equations 246 (10) (2009) 4129–4153.
[14] W. Strauss, C. Bu, An inhomogeneous boundary value problem for nonlinear Schrödinger equations, J. Differential Equations 173 (1) (2001) 79–91.
[15] X. Zong, Y. Zhao, Z. Yin, T. Chi, Exact boundary controllability of 1-D nonlinear Schrödinger equation, Appl. Math. J. Chinese Univ. Ser. B 22 (3) (2007) 277–285.